

THE FIRST AND SECOND FUNDAMENTAL THEOREMS OF INVARIANT THEORY FOR THE QUANTUM GENERAL LINEAR SUPERGROUP

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Dedicated to Professor Gus Lehrer on the occasion of his 70th birthday

ABSTRACT. We develop the non-commutative polynomial version of the invariant theory for the quantum general linear supergroup $U_q(\mathfrak{gl}_{m|n})$. A non-commutative $U_q(\mathfrak{gl}_{m|n})$ -module superalgebra $\mathcal{P}_{r|s}^{k|l}$ is constructed, which is the quantum analogue of the supersymmetric algebra over $\mathbb{C}^{k|l} \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{r|s} \otimes (\mathbb{C}^{m|n})^*$. We analyse the structure of the subalgebra of $U_q(\mathfrak{gl}_{m|n})$ -invariants in $\mathcal{P}_{r|s}^{k|l}$ by using the quantum super analogue of Howe duality.

The subalgebra of $U_q(\mathfrak{gl}_{m|n})$ -invariants in $\mathcal{P}_{r|s}^{k|l}$ is shown to be finitely generated. We determine its generators and establish a surjective superalgebra homomorphism from a braided supersymmetric algebra onto it. This establishes the first fundamental theorem of invariant theory for $U_q(\mathfrak{gl}_{m|n})$.

We show that the above mentioned superalgebra homomorphism is an isomorphism if and only if $m \geq \min\{k, r\}$ and $n \geq \min\{l, s\}$, and obtain a monomial basis for the subalgebra of invariants in this case. When the homomorphism is not injective, we give a representation theoretical description of the generating elements of the kernel associated to the partition $((m+1)^{n+1})$. This way we obtain the relations obeyed by the generators of the subalgebra of invariants, producing the second fundamental theorem of invariant theory for $U_q(\mathfrak{gl}_{m|n})$.

We consider two applications of our results. A complete treatment of the non-commutative polynomial version of invariant theory for $U_q(\mathfrak{gl}_m)$ is obtained as the special case with $n = 0$, where an explicit SFT is proved, which we believe to be new. The FFT and SFT of the invariant theory for the general linear superalgebra are recovered from the classical (i.e., $q \rightarrow 1$) limit of our results.

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1. INTRODUCTION

Let $\mathbb{C}^{m|n}$ be the natural module for the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$, and let $(\mathbb{C}^{m|n})^*$ be its dual. Denote by $\mathcal{S}_{r|s}^{k|l}$ the supersymmetric algebra over $\mathbb{C}^{k|l} \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{r|s} \otimes (\mathbb{C}^{m|n})^*$. One formulation of the invariant theory of $\mathfrak{gl}_{m|n}$ seeks to describe the subalgebra of $\mathfrak{gl}_{m|n}$ -invariants of $\mathcal{S}_{r|s}^{k|l}$. The first fundamental theorem (FFT) provides a finite set of generators for the subalgebra of invariants, and the second fundamental theorem (SFT) describes the relations among the generators (see [27, 28]).

The aim of this paper is to develop quantum analogues of FFT and SFT of invariant theory for the quantum general linear supergroup $U_q(\mathfrak{gl}_{m|n})$. More precisely, we construct a non-commutative analogue $\mathcal{P}_{r|s}^{k|l}$ of $\mathcal{S}_{r|s}^{k|l}$, which forms a module superalgebra over the quantum general linear supergroup. We investigate the $U_q(\mathfrak{gl}_{m|n})$ -action on $\mathcal{P}_{r|s}^{k|l}$, constructing a finite set of non-commutative generators for the subalgebra of invariants, and determining the algebraic relations obeyed by the generators. These results amount to an FFT and SFT, which are respectively given in Theorem 4.6 (also see Theorem 4.21) and Theorem 5.15.

The invariant theory of $U_q(\mathfrak{gl}_{m|n})$ in this non-commutative algebraic setting is poorly understood previously. One difficulty is to find an appropriate $\mathcal{P}_{r|s}^{k|l}$, which is flat in the sense of [2], that is, has $\mathcal{S}_{r|s}^{k|l}$ as classical limit ($q \rightarrow 1$). Fortunately in the present case, we can overcome the difficulty by using known results on coordinate superalgebras [36] of quantum general linear supergroups following ideas of [16]. Another difficulty arises from the non-semisimplicity of the $U_q(\mathfrak{gl}_{m|n})$ -action on $\mathcal{P}_{r|s}^{k|l}$. This makes it highly nontrivial to describe the invariants. We resolve this problem by making use the Howe duality of type $(U_q(\mathfrak{gl}_{k|l}), U_q(\mathfrak{gl}_{r|s}))$ given in Theorem 3.4 (see also [31, Theorem 2.2]).

Let us now describe in more details the main results of this paper.

We recall from [36] two bi-superalgebras $\mathcal{M}_{m|n}$ and $\overline{\mathcal{M}}_{m|n}$ of the finite dual of the quantum general linear supergroup $U_q(\mathfrak{gl}_{m|n})$, which are respectively generated by the matrix elements of the natural $U_q(\mathfrak{gl}_{m|n})$ -module $V^{m|n}$ and its dual $(V^{m|n})^*$. Particularly important to us is that both $\mathcal{M}_{m|n}$ and $\overline{\mathcal{M}}_{m|n}$ admit multiplicity-free decompositions as $U_q(\mathfrak{gl}_{m|n}) \otimes U_q(\mathfrak{gl}_{m|n})$ -module by a partial analogue of quantum Peter-Weyl theorem. By applying truncation procedures to them, we obtain in Section 3.1 superalgebras $\mathcal{M}_{r|s}^{k|l}$ and $\overline{\mathcal{M}}_{r|s}^{k|l}$ with $k, r \leq m$ and $l, s \leq n$, which are module superalgebras over $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$. They admit multiplicity-free decompositions as $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -modules, that is, there are Howe dualities of type $(U_q(\mathfrak{gl}_{k|l}), U_q(\mathfrak{gl}_{r|s}))$ on them; see Theorem 3.4 and Remark 3.5. We show that the Howe duality on $\mathcal{M}_{r|0}^{k|l}$ implies the quantum Schur-Weyl duality between $U_q(\mathfrak{gl}_{k|l})$ and the Hecke algebra $\mathcal{H}_q(r)$. Also by using the Howe duality, we construct a PBW basis (monomial basis) for $\mathcal{M}_{r|s}^{k|l}$.

The braided supersymmetric algebra in the sense of [2] provides the second formulation of $\mathcal{M}_{r|s}^{k|l}$ using the braiding operator on $V^{k|l} \otimes V^{r|s}$. This gives rise to a superalgebra isomorphism

between $\mathcal{M}_{r|s}^{k|l}$ and $S_q(V^{k|l} \otimes V^{r|s})$, where the latter is a quantum analogue of the supersymmetric algebra $S(\mathbb{C}^{k|l} \otimes \mathbb{C}^{r|s})$. From the Howe duality on $\mathcal{M}_{r|s}^{k|l}$, we deduce that $V^{k|l} \otimes V^{r|s}$ is a flat $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module, which means that $\dim_{\mathbb{C}(q)} S_q(V^{k|l} \otimes V^{r|s})_N = \dim_{\mathbb{C}} S(\mathbb{C}^{k|l} \otimes \mathbb{C}^{r|s})_N$ for any homogeneous component of degree $N \geq 0$. Similar discussions go through for $\widetilde{\mathcal{M}}_{r|s}^{k|l}$, which is isomorphic to the braided supersymmetric algebra $S_q((V^{k|l})^* \otimes (V^{r|s})^*)$. These results are described in Proposition 3.12 and Proposition 3.14.

To construct the non-commutative analogue of $\mathcal{S}_{r|s}^{k|l}$, we define $\mathcal{P}_{r|s}^{k|l} := \mathcal{M}_{m|n}^{k|l} \otimes_{\mathfrak{R}} \overline{\mathcal{M}}_{m|n}^{r|s}$, where k, l, r, s are any non-negative integers. This is still a $U_q(\mathfrak{gl}_{m|n})$ -module superalgebra with the multiplication rule given in Proposition 2.3. We denote by $\mathcal{X}_{r|s}^{k|l} := (\mathcal{P}_{r|s}^{k|l})^{U_q(\mathfrak{gl}_{m|n})}$ the subalgebra of $U_q(\mathfrak{gl}_{m|n})$ -invariants. The generators for $\mathcal{X}_{r|s}^{k|l}$ are constructed, along with their commutation relations. This amounts to the FFT of invariant theory, which is given in Theorem 4.6.

To fully explore the algebraic property of $\mathcal{X}_{r|s}^{k|l}$, we need a reformulation of FFT in terms of superalgebra homomorphism. We introduce an auxiliary quadratic superalgebra $\widetilde{\mathcal{M}}_{r|s}^{k|l}$, establishing a surjective superalgebra homomorphism $\Psi_{r|s}^{k|l} : \widetilde{\mathcal{M}}_{r|s}^{k|l} \rightarrow \mathcal{X}_{r|s}^{k|l}$ in Theorem 4.21. The auxiliary superalgebra $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ turns out to be isomorphic to the braided supersymmetric algebra $S_q(V^{k|l} \otimes (V^{r|s})^*)$. Thus $\Psi_{r|s}^{k|l}$ can be interpreted as the following surjective superalgebra homomorphism

$$S_q(V^{k|l} \otimes (V^{r|s})^*) \longrightarrow \left(S_q(V^{k|l} \otimes V^{m|n}) \otimes_{\mathfrak{R}} S_q((V^{r|s})^* \otimes (V^{m|n})^*) \right)^{U_q(\mathfrak{gl}_{m|n})}.$$

This coincides with the classical case [27, 28] in the limit $q \rightarrow 1$. We show that there exists Howe duality on $\widetilde{\mathcal{M}}_{r|s}^{k|l}$, which is isomorphic to $\mathcal{M}_{r|s}^{k|l}$ as $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module and has a PBW basis (monomial basis).

The SFT of invariant theory now seeks to describe the kernel of $\Psi_{r|s}^{k|l}$, as the images of nonzero elements in kernel give rise to relations among invariants. Let $K = \max\{k, r\}$ and $L = \max\{l, s\}$. We start by investigating the algebraic structure of $\mathcal{M}_{K|L}$, taking advantage of some nice properties of matrix elements. By a partial analogue of the Peter-Weyl theorem, $\mathcal{M}_{K|L}$ decomposes into a direct sum of subspaces T_λ , which are spanned by matrix elements of the simple tensor modules $L_\lambda^{K|L}$ for $U_q(\mathfrak{gl}_{K|L})$. We prove two important algebraical properties of $\mathcal{M}_{K|L}$:

- (1) (Proposition 5.2) The product of any two T_λ and T_μ admits a multiplicity-free decomposition into direct sum of certain T_γ 's;
- (2) (Theorem 5.5) The two-sided ideal in $\mathcal{M}_{K|L}$ generated by a single subspace T_λ admits a multiplicity-free decomposition into direct sum of T_γ over all (K, L) -hook partitions $\gamma \supseteq \lambda$.

We translate these results to the auxiliary superalgebra $\widetilde{\mathcal{M}}_{K|L} := \widetilde{\mathcal{M}}_{K|L}^{K|L}$. Note that $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ can be identified as a subalgebra of $\widetilde{\mathcal{M}}_{K|L}$. Our SFT of invariant theory asserts that $\text{Ker } \Psi_{r|s}^{k|l}$ as a two-sided ideal of $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ is generated by the subspace $\widetilde{T}_{\lambda_c} \cap \widetilde{\mathcal{M}}_{r|s}^{k|l}$ with $\lambda_c = ((m+1)^{n+1})$, where $\widetilde{T}_{\lambda_c} \subset \widetilde{\mathcal{M}}_{K|L}$ is an analogue of T_{λ_c} . In particular, we show that $\Psi_{r|s}^{k|l}$ is an isomorphism if and only if $m \geq \min\{k, r\}$ and $n \geq \min\{l, s\}$, and we give a PBW basis (monomial basis) for $\mathcal{X}_{r|s}^{k|l}$ in this case. These results are given in Theorem 5.15.

We consider two applications of our FFT and SFT of invariant theory for $U_q(\mathfrak{gl}_{m|n})$.

The quantum general linear group $U_q(\mathfrak{gl}_m)$ is a special case of $U_q(\mathfrak{gl}_{m|n})$ with $n = 0$. We immediately obtain the generators of the subalgebra of invariants (see Theorem 6.2), recovering the FFT of invariant theory given in [16, Theorem 6.10]. We prove that the kernel of the surjective algebra homomorphism is generated by all $(m+1) \times (m+1)$ quantum minors, see Theorem 6.3. This gives an explicit SFT, which appears to be new.

The universal enveloping algebra $U(\mathfrak{gl}_{m|n})$ is another extremal case, where $q \rightarrow 1$. We give a new proof of the known FFT and SFT in this case.

Related earlier work. There exist some earlier work on the non-commutative polynomial version of invariant theory for quantum groups. The FFT and SFT for the quantum symplectic group were discussed in [26], while a complete treatment of FFT for each quantum group associated with a classical Lie algebra was given in [16]. A conceptual advance in [16] is that it brings module algebras into the picture. The paper also gives a general method to construct non-commutative analogues of polynomial rings, making essential use of the braiding of quantum group arising from the universal \mathfrak{R} -matrix. However, to date, there is no complete treatment of SFTs for quantum groups. One of our results in this paper is the SFT for $U_q(\mathfrak{gl}_m)$, which may shed light on the SFTs for the quantum orthogonal and symplectic groups. The works [8, 9, 10] investigate the coinvariant theory for the quantum general linear (super) group, but our concept of ‘generators’ (FFT) and ‘relations’ (SFT) is different from theirs.

It was shown in [16, 31, 37] that (skew) Howe duality [11, 12] survives quantisation for the quantum general linear (super) group, and the resulting quantum Howe duality was applied to develop the q -deformed Segal-Shale-Weil representations. More recently, the quantum skew Howe dualities [3, 24, 23] were used in the categorification of representations of $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{gl}_{m|n})$. Here we extend the quantum Howe duality of type $(U_q(\mathfrak{gl}_{k|l}), U_q(\mathfrak{gl}_r))$ established in [31] to type $(U_q(\mathfrak{gl}_{k|l}), U_q(\mathfrak{gl}_{r|s}))$, and simplify the original proof in [31].

The paper [17] establishes a full tensor functor from the category of ribbon graphs to the category of finite dimensional representations of $U_q(\mathfrak{g})$ with $\mathfrak{g} = \mathfrak{gl}_{m|n}, \mathfrak{osp}(m|2n)$. This provides the FFT of invariant theory for the quantum supergroups in the endomorphism algebra setting. However, very little was known previously about the non-commutative polynomial version of invariant theory for quantum supergroups.

2. QUANTUM GENERAL LINEAR SUPERGROUP

We shall work over the field $\mathcal{K} := \mathbb{C}(q)$, where q is an indeterminate. For any vector superspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$, we let $[\] : A_{\bar{0}} \cup A_{\bar{1}} \longrightarrow \mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ be the parity map, that is, $[a] = \bar{i}$ if $a \in A_{\bar{i}}$. Tensor products of vector superspaces are again vector superspaces. We define the functorial permutation map

$$(2.1) \quad P : A \otimes B \longrightarrow B \otimes A$$

such that $a \otimes b \mapsto (-1)^{[a][b]} b \otimes a$ for homogeneous $a \in A$ and $b \in B$, and generalise to inhomogeneous elements linearly. If A is an associative superalgebra, we define the super bracket $[\] : A \otimes A \longrightarrow A$ such that $[X, Y] := XY - (-1)^{[X][Y]} YX$.

2.1. The quantum general linear supergroup $U_q(\mathfrak{gl}_{m|n})$. Denote $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. For any given $m, n \in \mathbb{Z}_+$, let $\mathbf{I}_{m|n} = \{1, 2, \dots, m+n\}$ and $\mathbf{I}'_{m|n} = \mathbf{I}_{m|n} \setminus \{m+n\}$. Let $\{\epsilon_a \mid a \in \mathbf{I}_{m|n}\}$ be the basis of a vector space with the non-degenerate symmetric bilinear form $(\epsilon_a, \epsilon_b) = (-1)^{[a]} \delta_{ab}$, where $[a] = \bar{0}$ if $a \leq m$ and $[a] = \bar{1}$ otherwise. The roots of the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ are $\epsilon_a - \epsilon_b$ for $a \neq b$ in $\mathbf{I}_{m|n}$.

Definition 2.1. ([35]) The quantum general linear supergroup $U_q(\mathfrak{gl}_{m|n})$ is the unital associative \mathcal{K} -superalgebra generated by the

$$\begin{aligned} &\text{even generators: } K_a, K_a^{-1}, E_{b,b+1}, E_{b+1,b}, \quad a, b \in \mathbf{I}_{m|n}, b \neq m, m+n, \\ &\text{and odd generators: } E_{m,m+1}, E_{m+1,m}, \end{aligned}$$

subject to the relations

$$\begin{aligned} (R1) \quad &K_a K_a^{-1} = K_a^{-1} K_a = 1, K_a K_b = K_b K_a; \\ (R2) \quad &K_a E_{b,b+1} K_a^{-1} = q^{(\epsilon_a, \epsilon_b - \epsilon_{b+1})} E_{b,b+1}; \\ (R3) \quad &[E_{a,a+1}, E_{b+1,b}] = \delta_{ab} \frac{K_a K_{a+1}^{-1} - K_a^{-1} K_{a+1}}{q_a - q_a^{-1}} \text{ with } q_a = q^{(\epsilon_a, \epsilon_a)} = q^{(-1)^{[a]}}; \\ (R4) \quad &E_{a,a+1} E_{b,b+1} = E_{b,b+1} E_{a,a+1}, \quad E_{a+1,a} E_{b+1,b} = E_{b+1,b} E_{a+1,a}, \quad a - b \geq 2; \end{aligned}$$

- (R5) $(E_{a,a+1})^2 E_{b,b+1} - (q + q^{-1}) E_{a,a+1} E_{b,b+1} E_{a,a+1} + E_{b,b+1} (E_{a,a+1})^2 = 0, \quad |a-b| = 1, a \neq m;$
 $(E_{a+1,a})^2 E_{b+1,b} - (q + q^{-1}) E_{a+1,a} E_{b+1,b} E_{a+1,a} + E_{b+1,b} (E_{a+1,a})^2 = 0, \quad |a-b| = 1, a \neq m;$
- (R6) $(E_{m,m+1})^2 = (E_{m+1,m})^2 = [E_{m-1,m+2}, E_{m,m+1}] = [E_{m+2,m-1}, E_{m+1,m}] = 0,$
 where $E_{m-1,m+2}$ and $E_{m+2,m-1}$ are determined recursively by

$$E_{a,b} = \begin{cases} E_{a,c} E_{c,b} - q_c^{-1} E_{c,b} E_{a,c}, & a < c < b, \\ E_{a,c} E_{c,b} - q_c E_{c,b} E_{a,c}, & b < c < a. \end{cases}$$

Note that $[E_{a,b}] = [a] + [b]$ for any quantum root vector $E_{a,b}$. It is well known that $U_q(\mathfrak{gl}_{m|n})$ has the structure of Hopf superalgebra with the following structural maps:

co-multiplication

$$\begin{aligned} \Delta(E_{a,a+1}) &= E_{a,a+1} \otimes K_a K_{a+1}^{-1} + 1 \otimes E_{a,a+1}, \\ \Delta(E_{a+1,a}) &= E_{a+1,a} \otimes 1 + K_a^{-1} K_{a+1} \otimes E_{a+1,a}, \\ \Delta(K_a^{\pm 1}) &= K_a^{\pm 1} \otimes K_a^{\pm 1}, \end{aligned}$$

co-unit

$$\epsilon(E_{a,a+1}) = \epsilon(E_{a+1,a}) = 0, \quad \forall a \in \mathbf{I}'_{m|n}, \quad \epsilon(K_b^{\pm 1}) = 1, \quad \forall b \in \mathbf{I}_{m|n},$$

and antipode

$$\begin{aligned} S(E_{a,a+1}) &= -E_{a,a+1} K_a^{-1} K_{a+1}, \\ S(E_{a+1,a}) &= -K_a K_{a+1}^{-1} E_{a+1,a}, \\ S(K_a^{\pm 1}) &= K_a^{\mp 1}. \end{aligned}$$

It should be pointed out that the antipode is a \mathbb{Z}_2 -graded algebra anti-automorphism, i.e., $S(xy) = (-1)^{[x][y]} S(y) S(x)$ for homogeneous $x, y \in U_q(\mathfrak{gl}_{m|n})$. We will use Sweedler's notation for the co-multiplication, i.e., $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ for all $x \in U_q(\mathfrak{gl}_{m|n})$.

2.2. Representations of $U_q(\mathfrak{gl}_{m|n})$. Throughout this paper, we will consider only $U_q(\mathfrak{gl}_{m|n})$ -modules which are \mathbb{Z}_2 -graded, locally finite and of type **1**. A module is locally finite if any vector v satisfies $\dim(U_q(\mathfrak{gl}_{m|n})v) < \infty$; a module is of type **1** if the K_a act semi-simply with eigenvalues q^k ($k \in \mathbb{Z}$).

The representation theory of $U_q(\mathfrak{gl}_{m|n})$ at generic q is quite similar to that of $\mathfrak{gl}(m|n)$, which was treated systematically in [35]. In Appendix A, we briefly discuss the representation theory for $U_q(\mathfrak{gl}_{m|n})$. Here we consider the tensor modules in detail following [36].

Let $V^{m|n}$ be the natural $U_q(\mathfrak{gl}_{m|n})$ -module, which has the standard basis $\{v_a \mid a \in \mathbf{I}_{m|n}\}$ such that $[v_a] = [a]$, and

$$K_a v_b = q_a^{\delta_{ab}} v_b, \quad E_{a,a\pm 1} v_b = \delta_{b,a\pm 1} v_a.$$

Denote by $\pi : U_q(\mathfrak{gl}_{m|n}) \rightarrow \text{End}(V^{m|n})$ the corresponding representation of $U_q(\mathfrak{gl}_{m|n})$ relative to the standard basis of $V^{m|n}$. Then

$$(2.2) \quad \begin{aligned} \pi(K_a^{\pm 1}) &= I + (q_a^{\pm 1} - 1) e_{aa}, \quad a \in \mathbf{I}_{m|n} \\ \pi(E_{b,b+1}) &= e_{b,b+1}, \quad \pi(E_{b+1,b}) = e_{b+1,b}, \quad b \in \mathbf{I}'_{m|n} \end{aligned}$$

where e_{ab} is the (a, b) -matrix unit of size $(m+n) \times (m+n)$.

Given $k \in \mathbb{Z}_+$, we consider the tensor power $(V^{m|n})^{\otimes k} = V^{m|n} \otimes \cdots \otimes V^{m|n}$ (k factors), which acquires a $U_q(\mathfrak{gl}_{m|n})$ -module structure through the iterated co-multiplication $\Delta^{(k-1)} = (\text{id} \times \Delta^{(k-2)})$ with $\Delta^{(1)} = \Delta$. It is proved in [36] that $(V^{m|n})^{\otimes k}$ is semi-simple for all k . To characterise their simple submodules, we define a subset $\Lambda_{m|n}$ of $\mathbb{Z}_+^{\times(m+n)}$ by

$$(2.3) \quad \Lambda_{m|n} = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{m+n}) \in \mathbb{Z}_+^{\times(m+n)} \mid \lambda_{m+1} \leq n, \lambda_a \geq \lambda_{a+1}, \forall a \in \mathbf{I}'_{m|n}\}.$$

Such λ is referred to as (m, n) -hook partition, and $|\lambda| = \sum_{a \in \mathbf{I}_{m|n}} \lambda_a$ is called the size of λ . We associate each $\lambda \in \Lambda_{m|n}$ with λ^\natural defined by

$$(2.4) \quad \lambda^\natural = \sum_{a=1}^m \lambda_a \epsilon_a + \sum_{b=1}^n \langle \lambda'_b - m \rangle \epsilon_{m+b} = (\lambda_1, \dots, \lambda_m; \langle \lambda'_1 - m \rangle, \dots, \langle \lambda'_n - m \rangle),$$

where $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ with $\lambda'_i = \#\{\lambda_j \mid \lambda_j \geq i, j \in \mathbf{I}_{m|n}\}$ is the transpose partition of λ , and $\langle u \rangle := \max\{u, 0\}$ for any integer u . Observe that $\lambda_{m+1} \leq n$ if and only if $\lambda'_{n+1} \leq m$. Now introduce the set $\Lambda_{m|n}^{\natural} = \{\lambda^{\natural} \mid \lambda \in \Lambda_{m|n}\}$, and the subsets $\Lambda_{m|n}^{\natural}(N) = \{\lambda^{\natural} \in \Lambda_{m|n}^{\natural} \mid |\lambda| = N\}$ for any $N \in \mathbb{Z}_+$.

We denote by $L_{\lambda}^{m|n}$ the irreducible $U_q(\mathfrak{gl}_{m|n})$ -module with highest weight $\lambda^{\natural} \in \Lambda^{\natural}$, that is, there exists a nonzero $v \in L_{\lambda}^{m|n}$, the highest weight vector, such that

$$E_{a,a+1}v = 0, \quad K_b v = q^{(\lambda^{\natural}, \epsilon_b)} v, \quad \forall a \in \mathbf{I}'_{m|n}, \quad b \in \mathbf{I}_{m|n}.$$

The following results are from [36].

Proposition 2.2. ([36])

- (1) Each $U_q(\mathfrak{gl}_{m|n})$ -module $(V^{m|n})^{\otimes N}$ with $N \in \mathbb{Z}_+$ can be decomposed into a direct sum of simple modules with highest weights belonging to $\Lambda_{m|n}^{\natural}(N)$.
- (2) Every irreducible $U_q(\mathfrak{gl}_{m|n})$ -module with highest weight belonging to $\Lambda_{m|n}^{\natural}(N)$ is contained in $(V^{m|n})^{\otimes N}$ as a simple submodule.

2.3. Module superalgebras. Recall that the Drinfeld version of $U_q(\mathfrak{gl}_{m|n})$ (defined over the power series ring $\mathbb{C}[[t]]$ with $q = \exp(t)$) admits a universal \mathfrak{R} matrix, which is an invertible even element in a completion of $U_q(\mathfrak{gl}_{m|n}) \otimes U_q(\mathfrak{gl}_{m|n})$ satisfying the relations

$$\begin{aligned} (2.5) \quad & \mathfrak{R}\Delta(x) = \Delta'(x)\mathfrak{R}, \quad \forall x \in U_q(\mathfrak{gl}_{m|n}), \\ (2.6) \quad & (\Delta \otimes \text{id})\mathfrak{R} = \mathfrak{R}_{13}\mathfrak{R}_{23}, \quad (\text{id} \otimes \Delta)\mathfrak{R} = \mathfrak{R}_{13}\mathfrak{R}_{12}, \end{aligned}$$

where Δ' is the opposite co-multiplication. It is well known that \mathfrak{R} satisfies the celebrated Yang-Baxter equation

$$\mathfrak{R}_{12}\mathfrak{R}_{13}\mathfrak{R}_{23} = \mathfrak{R}_{23}\mathfrak{R}_{13}\mathfrak{R}_{12}.$$

Let V and W be two $U_q(\mathfrak{gl}_{m|n})$ -modules. It follows from (2.5) that the map

$$\check{\mathfrak{R}} := P\mathfrak{R} : V \otimes W \rightarrow W \otimes V,$$

satisfies $\check{\mathfrak{R}}\Delta(x)(v \otimes w) = \Delta(x)\check{\mathfrak{R}}(v \otimes w)$, thus gives rise to an isomorphism of $U_q(\mathfrak{gl}_{m|n})$ -modules.

For the Jimbo version of the quantum general linear supergroup $U_q(\mathfrak{gl}_{m|n})$ (over \mathcal{K}) under consideration here, there exists a canonical braiding in the category of locally finite $U_q(\mathfrak{gl}_{m|n})$ -modules of type **1**, which plays the role of the universal \mathfrak{R} matrix.

In what follows, we shall make extensive use of module superalgebras over a Hopf superalgebra, which is a super analogue of module algebras in the sense of [21, §4.1], see also [16]. An associative superalgebra (A, μ) with multiplication μ and identity 1_A is a $U_q(\mathfrak{gl}_{m|n})$ -module superalgebra if A is $U_q(\mathfrak{gl}_{m|n})$ -module with

$$x \cdot \mu(a \otimes b) = \sum_{(x)} (-1)^{[x(2)][a]} \mu(x_{(1)} \cdot a \otimes x_{(2)} \cdot b), \quad x \cdot 1_A = \epsilon(x)1_A,$$

for homogeneous elements $a, b \in A$ and $x \in U_q(\mathfrak{gl}_{m|n})$.

Proposition 2.3. Let (A, μ_A) and (B, μ_B) be locally finite $U_q(\mathfrak{gl}_{m|n})$ -module superalgebras. Then $A \otimes B$ acquires a $U_q(\mathfrak{gl}_{m|n})$ -module superalgebra structure with the multiplication

$$\mu_{A,B} = (\mu_A \otimes \mu_B)(\text{id}_A \otimes \check{\mathfrak{R}} \otimes \text{id}_B),$$

where $\check{\mathfrak{R}} := P\mathfrak{R}$ with $P(a \otimes b) = (-1)^{[a][b]} b \otimes a$ for any homogeneous $a \in A, b \in B$.

Proof. Let $\mathfrak{R} = \sum_h \alpha_h \otimes \beta_h$. We have $[\alpha_h] = [\beta_h]$ for all h since \mathfrak{R} is an even element in $U_q(\mathfrak{gl}_{m|n}) \otimes U_q(\mathfrak{gl}_{m|n})$. Then under the given multiplication, it follows that

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = \sum_h (-1)^{([\alpha_h] + [b_1])[a_2] + [\alpha_h][\beta_h]} a_1(\beta_h \cdot a_2) \otimes (\alpha_h \cdot b_1)b_2$$

for any homogeneous $a_1, b_1, a_2, b_2 \in U_q(\mathfrak{gl}_{m|n})$. It is straightforward to check that for all homogeneous $x \in U_q(\mathfrak{gl}_{m|n})$,

$$x.((a_1 \otimes b_1)(a_2 \otimes b_2)) = \sum_{(x)} (-1)^{[x_{(2)}][[a_1]+[b_1]]} (x_{(1)}.(a_1 \otimes b_1))(x_{(2)}.(a_2 \otimes b_2)),$$

where we have used the identity $\check{\mathfrak{R}}\Delta(x)(a \otimes b) = \Delta(x)\check{\mathfrak{R}}(a \otimes b)$. \square

We write $A \otimes_{\mathfrak{R}} B$ for the braided tensor product of module superalgebras A and B as defined in Proposition 2.3 to distinguish it from the usual tensor product $A \otimes_{\mathcal{K}} B$.

Remark 2.4. Observe that the multiplication defined in this way is associative. Therefore, the $U_q(\mathfrak{gl}_{m|n})$ -module superalgebra structure on $A_1 \otimes_{\mathfrak{R}} A_2 \otimes_{\mathfrak{R}} \dots \otimes_{\mathfrak{R}} A_k$ is well-defined for any given $U_q(\mathfrak{gl}_{m|n})$ -module superalgebras A_i , $i = 1, 2, \dots, k$.

Let $A^{U_q(\mathfrak{gl}_{m|n})} = \{a \in A \mid x.a = \epsilon(x)a, \forall x \in U_q(\mathfrak{gl}_{m|n})\}$ be the subspace of $U_q(\mathfrak{gl}_{m|n})$ -invariants in the module superalgebra A . The following result is well known, see e.g. [16, Lemma 2.2].

Lemma 2.5. *Let A be a $U_q(\mathfrak{gl}_{m|n})$ -module superalgebra. Then $A^{U_q(\mathfrak{gl}_{m|n})}$ is a submodule superalgebra of A .*

Proof. For any homogeneous $a, b \in A^{U_q(\mathfrak{gl}_{m|n})}$ and $x \in U_q(\mathfrak{gl}_{m|n})$,

$$x.(ab) = \sum_{(x)} (-1)^{[x_{(2)}][a]} (x_{(1)}.a)(x_{(2)}.b) = \sum_{(x)} (-1)^{[x_{(2)}][a]} \epsilon(x_{(1)})\epsilon(x_{(2)})ab = \epsilon(x)ab,$$

where the last equation follows from the identity

$$\sum_{(x)} (-1)^{[x_{(2)}][a]} x_{(1)}\epsilon(x_{(2)}) = \sum_{(x)} x_{(1)}\epsilon(x_{(2)}) = x.$$

As a consequence, $ab \in A^{U_q(\mathfrak{gl}_{m|n})}$. \square

2.4. Quantised function algebras on $U_q(\mathfrak{gl}_{m|n})$.

2.4.1. The bi-superalgebra $\mathcal{M}_{m|n}$. Let

$$U_q(\mathfrak{gl}_{m|n})^\circ := \{f \in (U_q(\mathfrak{gl}_{m|n}))^* \mid \text{Ker } f \text{ contains a cofinite ideal of } U_q(\mathfrak{gl}_{m|n})\}$$

be the finite dual [21] of $U_q(\mathfrak{gl}_{m|n})$, which is a Hopf superalgebra with structure dualising that of $U_q(\mathfrak{gl}_{m|n})$. We have the matrix elements $t_{ab} \in U_q(\mathfrak{gl}_{m|n})^\circ$, $a, b \in \mathbf{I}_{m|n}$, defined by [36]

$$\langle t_{ab}, x \rangle = \pi(x)_{ab}, \quad \forall x \in U_q(\mathfrak{gl}_{m|n}),$$

where π is the natural representation defined as in (2.2). We set the \mathbb{Z}_2 -grading of t_{ab} by $[t_{ab}] = [a] + [b]$. Consider the subalgebra $\mathcal{M}_{m|n}$ of $U_q(\mathfrak{gl}_{m|n})^\circ$ generated by matrix elements t_{ab} ($a, b \in \mathbf{I}_{m|n}$). The multiplication which $\mathcal{M}_{m|n}$ inherits from $U_q(\mathfrak{gl}_{m|n})^\circ$ is given by

$$(2.7) \quad \langle tt', x \rangle = \sum_{(x)} \langle t \otimes t', x_{(1)} \otimes x_{(2)} \rangle = \sum_{(x)} (-1)^{[t'] [x_{(1)}]} \langle t, x_{(1)} \rangle \langle t', x_{(2)} \rangle$$

for all $t, t' \in \mathcal{M}_{m|n}$ and $x \in U_q(\mathfrak{gl}_{m|n})$.

We proceed to give the relations of the generators t_{ab} . Applying $\pi \otimes \pi$ to both sides of (2.5), we have

$$(2.8) \quad R(\pi \otimes \pi)\Delta(x) = (\pi \otimes \pi)\Delta'(x)R,$$

where $R := (\pi \otimes \pi)\mathfrak{R}$, which is of the form [36]

$$\begin{aligned} R &= q^{\sum_{a \in \mathbf{I}_{m|n}} (-1)^{[a]} e_{aa} \otimes e_{aa}} + (q - q^{-1}) \sum_{a < b} (-1)^{[b]} e_{ab} \otimes e_{ba} \\ &= I \otimes I + \sum_{a \in \mathbf{I}_{m|n}} (q_a - 1) e_{aa} \otimes e_{aa} + (q - q^{-1}) \sum_{a < b} (-1)^{[b]} e_{ab} \otimes e_{ba}, \end{aligned}$$

where $q_a = q^{(-1)^{[a]}}$. It is easy to directly verify that the matrix R satisfies the Yang-Baxter equation.

Write $R = \sum_{a,b,c,d \in \mathbf{I}_{m|n}} e_{ab} \otimes e_{cd} R_{b,d}^{a,c}$, then the nonzero entries of the matrix R are

$$R_{a,b}^{a,b} = 1, \quad a \neq b, \quad R_{a,a}^{a,a} = q_a, \quad R_{b,a}^{a,b} = q_b - q_b^{-1}, \quad a < b,$$

where $q_b - q_b^{-1} = (-1)^{[b]}(q - q^{-1})$ for all $a, b \in \mathbf{I}_{m|n}$. It follows from (2.8) that the generators t_{ab} obey relations

$$\sum_{a',b'} (-1)^{([a']+[c])([b]+[d])} R_{a'b',t_{a',c}t_{b',d}}^{a,b} = \sum_{a',b'} (-1)^{([a']+[c])([b]+[b'])} t_{bb'} t_{aa'} R_{c,d}^{a',b'},$$

which can be written more explicitly as

$$(2.9) \quad \begin{aligned} (t_{ab})^2 &= 0, & [a] + [b] &= \bar{1}, \\ t_{ac} t_{bc} &= (-1)^{([a]+[c])([b]+[c])} q_c t_{bc} t_{ac}, & a > b, \\ t_{ab} t_{ac} &= (-1)^{([a]+[c])([a]+[b])} q_a t_{ac} t_{ab}, & b > c, \\ t_{ac} t_{bd} &= (-1)^{([a]+[c])([b]+[d])} t_{bd} t_{ac}, & a > b, c < d, \\ t_{ac} t_{bd} &= (-1)^{([a]+[c])([b]+[d])} t_{bd} t_{ac} \\ &\quad + (-1)^{[a]([b]+[d])+[b][d]} (q - q^{-1}) t_{bc} t_{ad}, & a > b, c > d. \end{aligned}$$

It is worthy to note that $\mathcal{M}_{m|n}$ has a bi-superalgebra structure with the co-multiplication and the co-unit given by

$$\Delta(t_{ab}) = \sum_{c \in \mathbf{I}_{m|n}} (-1)^{([a]+[c])([c]+[b])} t_{ac} \otimes t_{cb}, \quad \epsilon(t_{ab}) = \delta_{ab}.$$

Remark 2.6. With the matrix notation $T = \sum_{a,b \in \mathbf{I}_{m|n}} e_{ab} \otimes t_{ab}$, relations (2.9) can be simply written in the form

$$RT_1 T_2 = T_2 T_1 R,$$

where $T_1 T_2 = \sum_{a,b,c,d \in \mathbf{I}_{m|n}} (-1)^{([a]+[b])([c]+[d])} e_{ab} \otimes e_{cd} \otimes t_{ab} t_{cd}$; see [36] and [25] for details.

There exist two kinds of actions on $\mathcal{M}_{m|n}$, corresponding to the left and right translations in the classical situation. We define the \mathcal{R} -action by

$$(2.10) \quad \mathcal{R} : \mathbf{U}_q(\mathfrak{gl}_{m|n}) \otimes \mathcal{M}_{m|n} \rightarrow \mathcal{M}_{m|n} \quad x \otimes f \mapsto \mathcal{R}_x(f) = \sum_{(f)} (-1)^{([f]+[x])[x]} f_{(1)} \langle f_{(2)}, x \rangle.$$

Recall that there is an anti-automorphism w of $\mathbf{U}_q(\mathfrak{gl}_{m|n})$ given by

$$w(E_{a,a+1}) = E_{a+1,a}, \quad w(E_{a+1,a}) = E_{a,a+1}, \quad w(K_a) = K_a.$$

This and the antipode S respectively give rise to another two left actions \mathcal{L} and $\tilde{\mathcal{L}}$

$$(2.11) \quad \begin{aligned} \mathcal{L} : \mathbf{U}_q(\mathfrak{gl}_{m|n}) \otimes \mathcal{M}_{m|n} &\rightarrow \mathcal{M}_{m|n} \quad x \otimes f \mapsto \sum_{(f)} \langle f_{(1)}, w(x) \rangle f_{(2)}, \\ \tilde{\mathcal{L}} : \mathbf{U}_q(\mathfrak{gl}_{m|n}) \otimes \mathcal{M}_{m|n} &\rightarrow \mathcal{M}_{m|n} \quad x \otimes f \mapsto \sum_{(f)} \langle f_{(1)}, S(x) \rangle f_{(2)}, \end{aligned}$$

The two actions \mathcal{L} (or $\tilde{\mathcal{L}}$) and \mathcal{R} graded-commute with each other, i.e.,

$$\mathcal{L}_x(\mathcal{R}_y(f)) = (-1)^{[x][y]} \mathcal{R}_x(\mathcal{L}_y(f)).$$

In addition, they preserve the algebraic structure of $\mathcal{M}_{m|n}$. Thus, $\mathcal{M}_{m|n}$ is a module superalgebra over $\mathcal{L}(\mathbf{U}_q(\mathfrak{gl}_{m|n})) \otimes \mathcal{R}(\mathbf{U}_q(\mathfrak{gl}_{m|n}))$. It admits the following multiplicity-free decomposition, which is a partial analogue of quantum Peter-Weyl theorem.

Theorem 2.7. ([36, Proposition 4]) *As an $\mathcal{L}(U_q(\mathfrak{gl}_{m|n})) \otimes \mathcal{R}(U_q(\mathfrak{gl}_{m|n}))$ -module,*

$$(2.12) \quad \mathcal{M}_{m|n} \cong \bigoplus_{\lambda \in \Lambda_{m|n}} L_\lambda^{m|n} \otimes L_\lambda^{m|n},$$

where $\Lambda_{m|n}$ is the set of (m, n) -hook partitions.

Remark 2.8. [36] $\mathcal{M}_{m|n} \cong \bigoplus_{\lambda \in \Lambda_{m|n}} (L_\lambda^{m|n})^* \otimes L_\lambda^{m|n}$ as $\tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{m|n})) \otimes \mathcal{R}(U_q(\mathfrak{gl}_{m|n}))$ -module.

2.4.2. *The Hopf superalgebra $\mathcal{K}[\mathrm{GL}_q(m|n)]$.* Following convention in [36], we define elements $\bar{t}_{ab} \in U_q(\mathfrak{gl}_{m|n})^\circ$ by

$$\langle \bar{t}_{ab}, x \rangle := \bar{\pi}(x)_{ab} = (-1)^{[b]([a]+[b])} \langle t_{ba}, S(x) \rangle, \quad \forall x \in U_q(\mathfrak{gl}_{m|n}), \quad a, b \in \mathbf{I}_{m|n}.$$

These are matrix elements of the dual vector representation $\bar{\pi}$ of $U_q(\mathfrak{gl}_{m|n})$ acting on $(V^{m|n})^*$. They generate a \mathbb{Z}_2 -graded bi-algebra $\overline{\mathcal{M}}_{m|n}$ with the multiplication as in (2.7), and co-multiplication Δ and co-unit ϵ given by

$$\Delta(\bar{t}_{ab}) = \sum_{c \in \mathbf{I}_{m|n}} (-1)^{([a]+[c])([c]+[b])} \bar{t}_{ac} \otimes \bar{t}_{cb} \quad \epsilon(\bar{t}_{ab}) = \delta_{ab}.$$

This is a $U_q(\mathfrak{gl}_{m|n})$ -module superalgebra with respect to left $U_q(\mathfrak{gl}_{m|n})$ -actions $\mathcal{L}, \tilde{\mathcal{L}}, \mathcal{R} : U_q(\mathfrak{gl}_{m|n}) \otimes \overline{\mathcal{M}}_{m|n} \rightarrow \overline{\mathcal{M}}_{m|n}$ similarly defined as in (2.10) and (2.11). Furthermore, $\overline{\mathcal{M}}_{m|n}$ admits the following decomposition as in Theorem 2.7.

Theorem 2.9. *As an $\tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{m|n})) \otimes \mathcal{R}(U_q(\mathfrak{gl}_{m|n}))$ -module,*

$$(2.13) \quad \overline{\mathcal{M}}_{m|n} \cong \bigoplus_{\lambda \in \Lambda_{m|n}} L_\lambda^{m|n} \otimes (L_\lambda^{m|n})^*.$$

Now we denote by $\mathcal{K}[\mathrm{GL}_q(m|n)]$ the subalgebra of $U_q(\mathfrak{gl}_{m|n})^\circ$ generated by all matrix elements t_{ab} and \bar{t}_{ab} . The relations, besides (2.9), can be derived similarly from [36]

$$(2.14) \quad (\bar{\pi} \otimes \bar{\pi})(\mathfrak{R}\Delta(x)) = (\bar{\pi} \otimes \bar{\pi})(\Delta'(x)\mathfrak{R}),$$

$$(2.15) \quad (\pi \otimes \bar{\pi})(\mathfrak{R}\Delta(x)) = (\pi \otimes \bar{\pi})(\Delta'(x)\mathfrak{R}),$$

where the second one enables us to factorise $\mathcal{K}[\mathrm{GL}_q(m|n)]$ into

$$\mathcal{K}[\mathrm{GL}_q(m|n)] = \mathcal{M}_{m|n} \overline{\mathcal{M}}_{m|n},$$

which inherits a natural bi-algebra structure from $\mathcal{M}_{m|n}$ and $\overline{\mathcal{M}}_{m|n}$. Straightforward calculations can verify the relation

$$\sum_{c \in \mathbf{I}_{m|n}} (-1)^{[a]([b]+[c])} t_{ac} \bar{t}_{bc} = \sum_{c \in \mathbf{I}_{m|n}} (-1)^{([a]+[c])([a]+[b]+[c])} \bar{t}_{ca} t_{cb} = \delta_{ab},$$

which implies that $\mathcal{K}[\mathrm{GL}_q(m|n)]$ is a Hopf superalgebra with the antipode S given by

$$(2.16) \quad S(t_{ab}) = (-1)^{[a]([a]+[b])} \bar{t}_{ba}, \quad S(\bar{t}_{ab}) = (-1)^{[b]([a]+[b])} q^{(2\rho, \epsilon_a - \epsilon_b)} t_{ba}.$$

Here $2\rho = \sum_{a < b} (-1)^{[a]+[b]} (\epsilon_a - \epsilon_b)$ and $\epsilon_a - \epsilon_b$ is a root for $\mathfrak{gl}_{m|n}$.

Remark 2.10. It is proved in [18, Theorem 3.5] that $\mathcal{K}[\mathrm{GL}_q(m|n)] \cong \mathcal{O}_q(\mathrm{GL}_{m|n})$ as Hopf superalgebras, where $\mathcal{O}_q(\mathrm{GL}_{m|n})$ is the coordinate superalgebra of quantum general linear supergroup.

Remark 2.11. Let $\bar{R} = (\bar{\pi} \otimes \bar{\pi})\mathfrak{R}$, by the formula in [36] we have

$$\begin{aligned} \bar{R} &= q^{\sum_{a \in \mathbf{I}_{m|n}} (-1)^{[a]} e_{aa} \otimes e_{aa}} + (q - q^{-1}) \sum_{a > b} (-1)^{[b]} e_{ab} \otimes e_{ba} \\ &= I \otimes I + \sum_{a \in \mathbf{I}_{m|n}} (q_a - 1) e_{aa} \otimes e_{aa} + (q - q^{-1}) \sum_{a > b} (-1)^{[b]} e_{ab} \otimes e_{ba}. \end{aligned}$$

Thus, relations (2.14) can be written more explicitly as follows,

$$\begin{aligned}
(\bar{t}_{ab})^2 &= 0, & [a] + [b] &= \bar{1}, \\
\bar{t}_{ac}\bar{t}_{bc} &= (-1)^{([a]+[c])([b]+[c])} q_c^{-1} \bar{t}_{bc}\bar{t}_{ac}, & a > b, \\
\bar{t}_{ab}\bar{t}_{ac} &= (-1)^{([a]+[c])([a]+[b])} q_a^{-1} \bar{t}_{ac}\bar{t}_{ab}, & b > c, \\
\bar{t}_{ac}\bar{t}_{bd} &= (-1)^{([a]+[c])([b]+[d])} \bar{t}_{bd}\bar{t}_{ac}, & a > b, c < d, \\
\bar{t}_{ac}\bar{t}_{bd} &= (-1)^{([a]+c)([b]+d)} \bar{t}_{bd}\bar{t}_{ac} \\
&\quad - (-1)^{[a]([b]+[d])+[b][d]} (q - q^{-1}) \bar{t}_{bc}\bar{t}_{ad}, & a > b, c > d.
\end{aligned}
\tag{2.17}$$

3. QUANTUM HOWE DUALITY OF TYPE $(U_q(\mathfrak{gl}_{k|l}), U_q(\mathfrak{gl}_{r|s}))$

3.1. Quantum super analogue of Howe duality. We adopt the following notation for convenience. If $\mathcal{G} = \{g_1, \dots, g_k\}$ is a finite set of elements in $U_q(\mathfrak{gl}_{m|n})$, we denote by $\langle \mathcal{G} \rangle$ the linear span of the elements $g_{i_1} g_{i_2} \dots g_{i_N}$ for all $N \in \mathbb{Z}_+$ and $1 \leq i_1, \dots, i_N \leq k$, that is, $\langle \mathcal{G} \rangle$ is subalgebra generated by \mathcal{G} . Given integers $1 \leq s, t \leq m+n$, we introduce the following subalgebra of $U_q(\mathfrak{gl}_{m|n})$

$$\begin{aligned}
\Upsilon_u &= \langle K_a \mid 1 \leq a \leq u \rangle, \quad 1 \leq u \leq m+n, \\
\overline{\Upsilon}_v &= \langle K_a \mid v+1 \leq a \leq m+n \rangle, \quad 0 \leq v \leq m+n-1.
\end{aligned}$$

For any $U_q(\mathfrak{gl}_{m|n})$ -module V , we define the Υ_u -invariant subspace of V by

$$V^{\Upsilon_u} := \{v \in V \mid K_a v = v, \forall 1 \leq a \leq u\},$$

and call this a truncation of V . The $\overline{\Upsilon}_v$ -invariant subspace $V^{\overline{\Upsilon}_v}$ can be defined similarly. The following two lemmas will be crucial for our proof of quantum Howe duality.

Lemma 3.1. *Let $\lambda \in \Lambda_{m|n}$ be an (m, n) -hook partition. The $\overline{\Upsilon}_v$ -invariant subspace of $L_\lambda^{m|n}$ is*

$$(L_\lambda^{m|n})^{\overline{\Upsilon}_v} \cong \begin{cases} L_\lambda^{m|(v-m)}, & \text{if } v > m \text{ and } \lambda_{m+1} \leq v-m, \\ L_\lambda^v, & \text{if } v \leq m \text{ and } \lambda_{v+1} = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $L_\lambda^{m|(v-m)}$ is the irreducible $U_q(\mathfrak{gl}_{m|v-m})$ -module with highest weight $(\lambda_1, \dots, \lambda_m; \langle \lambda'_1 - m \rangle, \dots, \langle \lambda'_{v-m} - m \rangle)$, and L_λ^v is the irreducible $U_q(\mathfrak{gl}_v)$ -module with highest weight $(\lambda_1, \dots, \lambda_v)$.

Proof. Note that $U_q(\mathfrak{gl}_{m|n-1})$ is canonically embedded in $U_q(\mathfrak{gl}_{m|n})$ as the subalgebra generated by the elements $E_{a,a+1}, E_{a+1,a}, K_b^{\pm 1}$ with $1 \leq a < m+n-1$ and $1 \leq b \leq m+n-1$. Thus we can consider the restriction of the $U_q(\mathfrak{gl}_{m|n})$ -module $L_\lambda^{m|n}$ to a module for $U_q(\mathfrak{gl}_{m|n-1})$. Let $v_\lambda \in L_\lambda^{m|n}$ be the highest weight vector for $U_q(\mathfrak{gl}_{m|n})$ with the highest weight λ^\natural (see (2.4)). Then $(L_\lambda^{m|n})_0 = U_q(\mathfrak{gl}_{m|n-1})v_\lambda$ forms an irreducible $U_q(\mathfrak{gl}_{m|n-1})$ -module. Note that K_{m+n} , which commutes with $U_q(\mathfrak{gl}_{m|n-1})$, acts on $(L_\lambda^{m|n})_0$ by multiplication by the scalar $q^{\langle \lambda'_n - m \rangle}$.

Denote by $U_q(\mathfrak{n}_-)_0$ the subalgebra of $U_q(\mathfrak{gl}_{m|n})$ generated by the elements $E_{m+n,a}$ with $1 \leq a \leq m+n-1$. Then $L_\lambda^{m|n} = U_q(\mathfrak{n}_-)_0(L_\lambda^{m|n})_0$. Thus, all the weights $\mu = (\mu_1, \mu_2, \dots, \mu_{m+n})$ of $L_\lambda^{m|n}$ satisfy $\mu_{m+n} \geq \langle \lambda'_n - m \rangle$. Therefore, $(L_\lambda^{m|n})^{K_{m+n}} = (L_\lambda^{m|n})^{\overline{\Upsilon}_{m+n-1}} = 0$ unless $\langle \lambda'_n - m \rangle = 0$, i.e., $\lambda'_n \leq m$, which is equivalent to the condition that $\lambda_{m+1} \leq n-1$ (recall that $\lambda_{k+1} \leq l \Leftrightarrow \lambda'_{l+1} \leq k$ for any $k, l \in \mathbb{Z}_+$). In this case, we have $(L_\lambda^{m|n})^{\overline{\Upsilon}_{m+n-1}} = (L_\lambda^{m|n})_0$, which is an irreducible $U_q(\mathfrak{gl}_{m|n-1})$ -module with the highest weight $(\lambda_1, \dots, \lambda_m; \langle \lambda'_1 - m \rangle, \dots, \langle \lambda'_{n-1} - m \rangle)$.

Iterating this truncation procedure for $(L_\lambda^{m|n})^{\overline{\Upsilon}_{m+n-1}}$, we arrive at the assertion. \square

Lemma 3.2. *Let $\lambda \in \Lambda_{m|n}$ be an (m, n) -hook partition. The Υ_u -invariant subspace of $L_\lambda^{m|n}$ is*

$$(L_\lambda^{m|n})^{\Upsilon_u} \cong \begin{cases} L_\lambda^{(m-u)|n}, & \text{if } u < m \text{ and } \lambda_{m-u+1} \leq n, \\ L_\lambda^{m+n-u}, & \text{if } u \geq m \text{ and } \lambda'_{m+n-u+1} = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $L_\lambda^{m-u|n}$ is the irreducible $U_q(\mathfrak{gl}_{m-u|n})$ -module with highest weight $(\lambda_1, \dots, \lambda_{m-u}; \langle \lambda'_1 - (m-u) \rangle, \dots, \langle \lambda'_n - (m-u) \rangle)$, and $L_{\lambda'}^{m+n-u}$ is the irreducible $U_q(\mathfrak{gl}_{m+n-u})$ -module with highest weight $(\lambda'_1, \dots, \lambda'_{m+n-u})$.

Proof. Let $\bar{\lambda}^\natural$ be the lowest weight of $L_\lambda^{m|n}$. By Proposition A.4 in Appendix A, we have

$$\bar{\lambda}^\natural = (\langle \lambda_m - n \rangle, \langle \lambda_{m-1} - n \rangle, \dots, \langle \lambda_1 - n \rangle; \lambda'_n, \lambda'_{n-1}, \dots, \lambda'_1).$$

For convenience, we write $\Gamma_{\bar{\lambda}}^{m|n}$ for $L_{\bar{\lambda}}^{m|n}$ and $v_{\bar{\lambda}}$ for the lowest weight vector of $\Gamma_{\bar{\lambda}}^{m|n}$. Then $(\Gamma_{\bar{\lambda}}^{m|n})_0 := U_q(\mathfrak{gl}_{m-1|n})v_{\bar{\lambda}}$ is an irreducible $U_q(\mathfrak{gl}_{m-1|n})$ -module, where $U_q(\mathfrak{gl}_{m-1|n})$ is canonically embedded in $U_q(\mathfrak{gl}_{m|n})$ as the subalgebra generated by the elements $E_{a,a+1}, E_{a+1,a}, K_b^{\pm 1}$ with $2 \leq a \leq m+n-1$ and $2 \leq b \leq m+n$. Now K_1 , which commutes with $U_q(\mathfrak{gl}_{m-1|n})$, acts on $(L_\lambda^{m|n})_0$ by multiplication by the scalar $q^{\langle \lambda_m - n \rangle}$.

Denote by $U_q(\mathfrak{n}_+)_0$ the subalgebra of $U_q(\mathfrak{gl}_{m|n})$ generated by $E_{1,a}, 2 \leq a \leq m+n$, it follows that $\Gamma_{\bar{\lambda}}^{m|n} = U_q(\mathfrak{n}_+)_0(\Gamma_{\bar{\lambda}}^{m|n})_0$. Thus, all the weights $\mu = (\mu_1, \mu_2, \dots, \mu_{m+n})$ of $\Gamma_{\bar{\lambda}}^{m|n}$ satisfy $\mu_1 \geq \langle \lambda_m - n \rangle$. Therefore, it is clear that $(\Gamma_{\bar{\lambda}}^{m|n})^{K_1} = (\Gamma_{\bar{\lambda}}^{m|n})^{\Upsilon_1} = 0$ unless $\langle \lambda_m - n \rangle = 0$, i.e., $\lambda_m \leq n$. In this case, we have $(\Gamma_{\bar{\lambda}}^{m|n})^{\Upsilon_1} = (\Gamma_{\bar{\lambda}}^{m|n})_0$, which is an irreducible $U_q(\mathfrak{gl}_{m-1|n})$ -module with the lowest weight $(\langle \lambda_{m-1} - n \rangle, \dots, \langle \lambda_1 - n \rangle; \lambda'_n, \lambda'_{n-1}, \dots, \lambda'_1)$. This by Proposition A.4 means that $(\Gamma_{\bar{\lambda}}^{m|n})^{\Upsilon_1} = L_\lambda^{(m-1)|n}$ has the highest weight $(\lambda_1, \dots, \lambda_{m-1}; \langle \lambda'_1 - (m-1) \rangle, \dots, \langle \lambda'_n - (m-1) \rangle)$. Iterations of this truncation procedure lead to the assertion. \square

Given integers k, l that $0 \leq k \leq m-1$ and $0 \leq l \leq n-1$, we define a new subalgebra of $U_q(\mathfrak{gl}_{m|n})$ by

$$(3.1) \quad \Upsilon_{k|l} := \Upsilon_{m-k} \bar{\Upsilon}_{m+l} = \langle K_a \mid a \in \{1, \dots, m-k\} \cup \{m+l+1, \dots, m+n\} \rangle,$$

and then define the following truncation of $\mathcal{M}_{m|n}$:

$$(3.2) \quad \mathcal{M}_{r|s}^{k|l} := (\mathcal{M}_{m|n})^{\mathcal{L}(\Upsilon_{k|l}) \otimes \mathcal{R}(\Upsilon_{r|s})} = \{f \in \mathcal{M}_{m|n} \mid \mathcal{L}_{\Upsilon_{k|l}}(f) = \mathcal{R}_{\Upsilon_{r|s}}(f) = f\}.$$

Introduce the set $\hat{\mathbf{I}}_{k|l} := \{a \mid m-k+1 \leq a \leq m+l\}$, and similarly the set $\hat{\mathbf{I}}_{r|s}$. Then the elements $\{K_a, E_{b,b+1}, E_{b+1,b} \mid a \in \hat{\mathbf{I}}_{k|l}, b \in \hat{\mathbf{I}}_{k|l}, b \neq m+l\}$ generate a Hopf subalgebra $U_q(\mathfrak{gl}_{k|l})$ of $U_q(\mathfrak{gl}_{m|n})$. Clearly, $\Upsilon_{k|l}$ commutes with the subalgebra $U_q(\mathfrak{gl}_{k|l})$.

We obtain the following presentation for $\mathcal{M}_{r|s}^{k|l}$.

Lemma 3.3. *The subalgebra $\mathcal{M}_{r|s}^{k|l}$ of $\mathcal{M}_{m|n}$ is generated by $\{t_{ab} \mid a \in \hat{\mathbf{I}}_{k|l}, b \in \hat{\mathbf{I}}_{r|s}\}$ subject to the relevant relations of (2.9).*

Proof. Let $\prod_{k|l} = \prod_{a=1}^{m-k} \prod_{b=m+l+1}^{m+n} K_a K_b$, and similarly introduce $\prod_{r|s}$. Observe that for any $t_{ab} \in \mathcal{M}_{m|n}$

$$\mathcal{L}_{\prod_{k|l}}(t_{ab}) = \begin{cases} t_{ab}, & a \in \hat{\mathbf{I}}_{k|l}, \\ q_a t_{ab}, & \text{otherwise,} \end{cases} \quad \mathcal{R}_{\prod_{r|s}}(t_{ab}) = \begin{cases} t_{ab}, & a \in \hat{\mathbf{I}}_{r|s}, \\ q_b t_{ab}, & \text{otherwise.} \end{cases}$$

Therefore, $t_{a_1, b_1} t_{a_2, b_2} \dots t_{a_p, b_p} \in \mathcal{M}_{r|s}^{k|l}$ if and only if $a_i \in \hat{\mathbf{I}}_{k|l}$ and $b_i \in \hat{\mathbf{I}}_{r|s}$ for all $1 \leq i \leq p$. This implies that $\mathcal{M}_{r|s}^{k|l}$ is generated by $\{t_{ab} \mid a \in \hat{\mathbf{I}}_{k|l}, b \in \hat{\mathbf{I}}_{r|s}\}$, while the relations follows directly from (2.9). \square

The following theorem is the quantum Howe duality of type $(U_q(\mathfrak{gl}_{k|l}), U_q(\mathfrak{gl}_{r|s}))$ applied to the subalgebra $\mathcal{M}_{r|s}^{k|l}$ of $\mathcal{M}_{m|n}$ with $k, r \leq m$ and $l, s \leq n$.

Theorem 3.4. *(($U_q(\mathfrak{gl}_{k|l}), U_q(\mathfrak{gl}_{r|s})$)-duality) The superalgebra $\mathcal{M}_{r|s}^{k|l}$ admits a multiplicity-free decomposition as $\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \mathcal{R}(U_q(\mathfrak{gl}_{r|s}))$ -module*

$$(3.3) \quad \mathcal{M}_{r|s}^{k|l} \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} L_\lambda^{k|l} \otimes L_\lambda^{r|s},$$

where $\Lambda_{k|l}$ (resp. $\Lambda_{r|s}$) is the set of $(k|l)$ (resp. $(r|s)$) hook partitions.

Proof. By Theorem 2.7, we have

$$\mathcal{M}_{r|s}^{k|l} = (\mathcal{M}_{m|n})^{\mathcal{L}(\Upsilon_{k|l}) \otimes \mathcal{R}(\Upsilon_{r|s})} \cong \bigoplus_{\lambda \in \Lambda_{m|n}} (L_\lambda^{m|n})^{\Upsilon_{k|l}} \otimes (L_\lambda^{m|n})^{\Upsilon_{r|s}}.$$

By Lemma 3.1 and Lemma 3.2,

$$(L_\lambda^{m|n})^{\Upsilon_{k|l}} = ((L_\lambda^{m|n})^{\Upsilon_{m-k}})^{\Upsilon_{m+l}} \cong (L_\lambda^{k|n})^{\Upsilon_{m+l}} \cong L_\lambda^{k|l},$$

where the second isomorphism requires $\lambda_{k+1} \leq n$ and the last one $\lambda_{k+1} \leq l$, yielding $\lambda \in \Lambda_{k|l}$. The same argument applies to $(L_\lambda^{m|n})^{\Upsilon_{r|s}}$. Thus, $(L_\lambda^{m|n})^{\Upsilon_{k|l}} \otimes (L_\lambda^{m|n})^{\Upsilon_{r|s}} \cong L_\lambda^{k|l} \otimes L_\lambda^{r|s}$ if $\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}$, and it will be 0 otherwise. \square

Remark 3.5. Similarly, we can show that as $\tilde{\mathcal{L}}(\mathrm{U}_q(\mathfrak{gl}_{k|l})) \otimes \mathcal{R}(\mathrm{U}_q(\mathfrak{gl}_{r|s}))$ -module,

$$\overline{\mathcal{M}}_{r|s}^{k|l} \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} L_\lambda^{k|l} \otimes (L_\lambda^{r|s})^*.$$

Notation. We shall regard $\mathcal{M}_{r|s}^{k|l}$ as the superalgebra generated by t_{ab} with $a \in \mathbf{I}_{k|l}$ and $b \in \mathbf{I}_{r|s}$ subject to relations of the form (2.9), as the two superalgebras are isomorphic.

Remark 3.6. We have following facts in the classical case ($q \rightarrow 1$):

- (1) By specialising q to 1, relations (2.9) reduce to the supercommutative relations $[t_{ab}, t_{cd}] = 0$. In this case, we denote by $\mathcal{M}_{r|s}^{k|l}|_{q=1}$ the superalgebra over \mathbb{C} generated by t_{ab} . This superalgebra is isomorphic to the supersymmetric algebra $S(\mathbb{C}^{k|l} \otimes \mathbb{C}^{r|s})$, where $\mathbb{C}^{k|l}$ and $\mathbb{C}^{r|s}$ are respectively natural representations of $\mathfrak{gl}_{k|l}$ and $\mathfrak{gl}_{r|s}$. Thus, Theorem 3.4 recovers classical Howe duality for $S(\mathbb{C}^{k|l} \otimes \mathbb{C}^{r|s})$ (see [4, Theorem 3.2] and [28, Theorem 1.3]), that is, the decomposition (3.3) holds for $\mathcal{M}_{r|s}^{k|l}|_{q=1}$ in the case of classical limit $q \rightarrow 1$. In general, we will show that $\mathcal{M}_{r|s}^{k|l}$ is isomorphic to a quantum analogue of supersymmetric algebra; see Section 3.2.
- (2) For any $k, l \in \mathbb{Z}_+$, we denote by $L_\lambda^{k|l}|_{q=1}$ the irreducible $\mathfrak{gl}_{k|l}$ -module corresponding to the irreducible $\mathrm{U}_q(\mathfrak{gl}_{k|l})$ -module $L_\lambda^{k|l}$. It was proved in [35, Proposition 3] that $\dim_{\mathbb{C}} L_\lambda^{k|l}|_{q=1} = \dim_{\mathbb{C}} L_\lambda^{k|l}$. This will be used frequently in what follows.

As a quick application of the quantum Howe duality, we obtain an explicit PBW basis for $\mathcal{M}_{r|s}^{k|l}$, which is a special case of [20, Theorem 1.14]. Let $\mathbf{m} = (m_{ab})$, $a \in \mathbf{I}_{k|l}$, $b \in \mathbf{I}_{r|s}$ be a $(k+l) \times (r+s)$ supermatrix with parity assignment $[m_{ab}] = [a] + [b]$ such that $m_{ab} \in \mathbb{Z}_+$ whenever $[m_{ab}] = \bar{0}$ and $m_{ab} \in \{0, 1\}$ whenever $[m_{ab}] = \bar{1}$. We denote by $\mathfrak{M}_{r|s}^{k|l}$ the set of all such supermatrices \mathbf{m} . Introduce a linear order $>$ for the pairs (a, b) such that

$$(3.4) \quad (a, b) > (c, b+k), \quad (a, b) > (a+k, b), \quad \forall k > 0.$$

Define monomial $t^{\mathbf{m}} \in \mathcal{M}_{r|s}^{k|l}$ by

$$t^{\mathbf{m}} = \prod_{(a,b)}^> t_{ab}^{m_{ab}} = t_{11}^{m_{11}} t_{21}^{m_{21}} \cdots t_{12}^{m_{12}} t_{22}^{m_{22}} \cdots, \quad \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l},$$

where the factors are arranged decreasingly in the order $>$. Note that $\mathcal{M}_{r|s}^{k|l}$ is \mathbb{Z}_+ -graded with the gradation $\deg t_{ab} = 1$. We write $|\mathbf{m}| := \sum_{a,b} m_{ab}$ for the degree of monomial $t^{\mathbf{m}}$, and denote by $(\mathcal{M}_{r|s}^{k|l})_N$ the homogeneous subspace of degree N in $\mathcal{M}_{r|s}^{k|l}$.

Lemma 3.7. *We have the following identities:*

$$\begin{aligned} \sum_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}, |\lambda|=N} \dim_{\mathcal{K}} L_{\lambda}^{k|l} \dim_{\mathcal{K}} L_{\lambda}^{r|s} &= \sum_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}, |\lambda|=N} \dim_{\mathbb{C}} L_{\lambda}^{k|l}|_{q=1} \dim_{\mathbb{C}} L_{\lambda}^{r|s}|_{q=1} \\ &= \#\{t^{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}, |\mathbf{m}| = N\}, \end{aligned}$$

where $\#S$ stands for the cardinality of set S .

Proof. The first identity follows from Remark 3.6, so we only need to prove the second one. It is easy to see that $t^{\mathbf{m}}, \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}$ form a basis for $\mathcal{M}_{r|s}^{k|l}|_{q=1}$, which is isomorphic to the super polynomial algebra generated by $kr + ls$ even variables and $kl + rs$ Grassmannian variables (cf. [29, Theorem 3.1]). Let $(\mathcal{M}_{r|s}^{k|l}|_{q=1})_N$ be the homogeneous subspace of degree N in $\mathcal{M}_{r|s}^{k|l}|_{q=1}$. Then we obtain

$$\dim_{\mathbb{C}}(\mathcal{M}_{r|s}^{k|l}|_{q=1})_N = \#\{t^{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}, |\mathbf{m}| = N\}.$$

On the other hand, it follows from classical Howe duality for $\mathcal{M}_{r|s}^{k|l}|_{q=1}$ that

$$\dim_{\mathbb{C}}(\mathcal{M}_{r|s}^{k|l}|_{q=1})_N = \sum_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}, |\lambda|=N} \dim_{\mathbb{C}} L_{\lambda}^{k|l}|_{q=1} \dim_{\mathbb{C}} L_{\lambda}^{r|s}|_{q=1}.$$

This proves the second identity. \square

We obtain the following PBW basis for superalgebra $\mathcal{M}_{r|s}^{k|l}$.

Proposition 3.8. *The set of monomials $\{t^{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}\}$ constitutes a \mathcal{K} -basis for $\mathcal{M}_{r|s}^{k|l}$.*

Proof. We deduce from relations (2.9) that $\mathcal{M}_{r|s}^{k|l}$ is spanned by the set of given monomials, and hence it remains to show the linear independence. By Theorem 3.4, we have

$$(3.5) \quad (\mathcal{M}_{r|s}^{k|l})_N \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}, |\lambda|=N} L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|s},$$

which implies that

$$\dim_{\mathcal{K}}(\mathcal{M}_{r|s}^{k|l})_N = \sum_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}, |\lambda|=N} \dim_{\mathcal{K}} L_{\lambda}^{k|l} \dim_{\mathcal{K}} L_{\lambda}^{r|s}.$$

Combining this and Lemma 3.7, we obtain that

$$\dim_{\mathcal{K}}(\mathcal{M}_{r|s}^{k|l})_N = \#\{t^{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}, |\mathbf{m}| = N\}.$$

Since the monomials $t^{\mathbf{m}}$ with $|\mathbf{m}| = N$ span the degree N homogeneous subspace $(\mathcal{M}_{r|s}^{k|l})_N$, they must be linearly independent. This completes our proof. \square

We now turn to another formulation of $\mathcal{M}_{r|s}^{k|l}$.

3.2. Braided supersymmetric algebra and flat modules. Let V be a finite dimensional module over $U_q(\mathfrak{gl}_{m|n})$, and $R_{V,V} \in \text{GL}(V \otimes V)$ be the associated R -matrix. Recall that $\check{R}_{V,V} = PR_{V,V}$, where P is the graded permutation map (2.1). Then $\check{R}_{V,V} \in \text{End}_{U_q(\mathfrak{gl}_{m|n})}(V \otimes V)$ with

$$(3.6) \quad (\check{R}_{V,V} \otimes \text{id}_V)(\text{id}_V \otimes \check{R}_{V,V})(\check{R}_{V,V} \otimes \text{id}_V) = (\text{id}_V \otimes \check{R}_{V,V})(\check{R}_{V,V} \otimes \text{id}_V)(\text{id}_V \otimes \check{R}_{V,V}).$$

It is well-known that $\check{R}_{V,V}$ acts on $V \otimes V$ semi-simply, and its eigenvalues are of the form $\pm q^r$ with $r \in \frac{1}{2}\mathbb{Z}$. Thus $\check{R}_{V,V}$ obeys a minimal characteristic polynomial of the form

$$\prod_{i=1}^{k^+} (\check{R}_{V,V} - q^{x_i^+}) \prod_{i=1}^{k^-} (\check{R}_{V,V} + q^{x_i^-}) = 0,$$

where $k^\pm \in \mathbb{Z}_+$ and $\chi_i^+, \chi_i^- \in \frac{1}{2}\mathbb{Z}$. Define the following submodules of $V \otimes V$

$$S_q^2(V) = \{w \in V \otimes V \mid \prod_{i=1}^{k^+} (\check{R}_{V,V} - q^{\chi_i^+})(w) = 0\},$$

$$\Lambda_q^2(V) = \{w \in V \otimes V \mid \prod_{i=1}^{k^-} (\check{R}_{V,V} + q^{\chi_i^-})(w) = 0\}.$$

Definition 3.9. ([2]) Let V be \mathbb{Z}_2 -graded finite-dimensional $U_q(\mathfrak{gl}_{m|n})$ -module. We define the braided supersymmetric algebra and braided superexterior algebra respectively by

$$S_q(V) = T(V)/\langle \Lambda_q^2(V) \rangle, \quad \Lambda_q(V) = T(V)/\langle S_q^2(V) \rangle,$$

where $T(V)$ is the tensor superalgebra of V and $\langle I \rangle$ denotes the 2-sided ideal generated by subset $I \subset T(V)$.

Let $V|_{q=1}$ be the complex vector space spanned by the same basis elements as V . The supersymmetric algebra $S(V|_{q=1})$ over \mathbb{C} is \mathbb{Z}_+ -graded, so is $S_q(V)$. As a quotient module of $T(V)$, $S_q(V)$ encodes $U_q(\mathfrak{gl}_{m|n})$ -module structure which preserves the algebraic structure, and hence it is a module superalgebra.

Following [2], we will call V *flat module* if and only if $\dim_{\mathcal{K}} S_q(V)_N = \dim_{\mathbb{C}} S(V|_{q=1})_N$ for all $N \in \mathbb{Z}_+$, and $S_q(V)$ is called the *flat deformation* of $S(V|_{q=1})$. The flat deformation $S_q(V)$ acts as quantum analogue of supersymmetric algebra. However, other than natural modules, flat modules are extremely rare even in the quantum group case; see [16, §2.2].

We now give concrete examples of flat modules. Let $V^{k|l}$ be the natural representation for $U_q(\mathfrak{gl}_{k|l})$ with the standard weight basis $\{v_i\}_{i \in \mathbf{I}_{k|l}}$. Then $\check{R}_{V^{k|l}, V^{k|l}}$ gives an automorphism of $V^{k|l} \otimes V^{k|l}$ with the action

$$\check{R}_{V^{k|l}, V^{k|l}}(v_i \otimes v_j) = \begin{cases} (-1)^{[i][j]} v_j \otimes v_i, & \text{if } i < j, \\ (-1)^{[i]} q_i v_i \otimes v_i, & \text{if } i = j, \\ (-1)^{[i][j]} v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j, & \text{if } i > j. \end{cases}$$

The tensor product $V^{k|l} \otimes V^{k|l}$ can be decomposed as direct sum of two irreducible $U_q(\mathfrak{gl}_{k|l})$ -submodules,

$$V^{k|l} \otimes V^{k|l} = S_q^2(V^{k|l}) \oplus \Lambda_q^2(V^{k|l}).$$

The basis for $S_q^2(V^{k|l})$ is given by

$$(3.7) \quad v_i \otimes v_i, \quad 1 \leq i \leq k, \quad v_i \otimes v_j + (-1)^{[i][j]} q v_j \otimes v_i, \quad 1 \leq i < j \leq k+l,$$

while the basis for $\Lambda_q^2(V^{k|l})$ is

$$(3.8) \quad v_i \otimes v_i, \quad k+1 \leq i \leq k+l, \quad v_i \otimes v_j - (-1)^{[i][j]} q^{-1} v_j \otimes v_i, \quad 1 \leq i < j \leq k+l.$$

Let $P_s^{(k|l)}$ and $P_a^{(k|l)}$ be the idempotent projection onto the irreducible submodules $S_q^2(V^{k|l})$ and $\Lambda_q^2(V^{k|l})$, respectively. Then we have $\check{R}_{V^{k|l}, V^{k|l}} = qP_s^{(k|l)} - q^{-1}P_a^{(k|l)}$, which leads to

$$(3.9) \quad (\check{R}_{V^{k|l}, V^{k|l}} - q)(\check{R}_{V^{k|l}, V^{k|l}} + q^{-1}) = 0.$$

Proposition 3.10. Let $k, l \in \mathbb{Z}_+$, we have

- (1) As a superalgebra, $\mathcal{M}_{1|0}^{k|l} \cong S_q(V^{k|l})$, which is generated by x_1, x_2, \dots, x_{k+l} with parity assignments $[x_i] = [i]$ and defining relations

$$x_i^2 = 0, \quad \text{if } [i] = \bar{1},$$

$$x_j x_i = (-1)^{[i][j]} q x_i x_j \quad 1 \leq i < j \leq k+l.$$

- (2) The natural representation $V^{k|l}$ is flat, and $S_q(V^{k|l}) = \bigoplus_{N \in \mathbb{Z}_+} S_q(V^{k|l})_N$, where $S_q(V^{k|l})_N$ is the irreducible $U_q(\mathfrak{gl}_{k|l})$ -module with the highest weight $N\epsilon_1$.

Proof. The relations for $S_q(V^{k|l})$ in Part (1) come from (3.8), and the isomorphism is given by $t_{i1} \mapsto x_i$ for $1 \leq i \leq k+l$. By Theorem 3.4, as $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{1|0}) \cong U_q(\mathfrak{gl}_{k|l})$ module,

$$S_q(V^{k|l}) \cong \mathcal{M}_{1|0}^{k|l} \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{1|0}} L_\lambda^{k|l} \otimes_{\mathcal{K}} \mathcal{K},$$

where the sum is over all one-row Young diagrams. For any one-row Young diagram λ with N boxes, we obtain $S_q(V^{k|l})_N \cong L_\lambda^{k|l}$ and

$$\dim_{\mathcal{K}} S_q(V^{k|l})_N = \dim_{\mathcal{K}} L_\lambda^{k|l} = \dim_{\mathbb{C}} L_\lambda^{k|l}|_{q=1} = \dim_{\mathbb{C}} S(\mathbb{C}^{k|l})_N.$$

Thus, $V^{k|l}$ is a flat $U_q(\mathfrak{gl}_{k|l})$ -module. \square

The following proposition will not be used later, but is interesting in its own right. Recall that Manin [20] introduced two multiparameter quantum superspaces A_q and A_q^* of superdimensions $(k|l)$. In the one-parameter case $A_q \cong \mathcal{M}_{1|0}^{k|l}$ as superalgebras, while $A_q^* \cong \mathcal{M}_{0|1}^{k|l}$ as shown in the following proposition.

Proposition 3.11. *Let $k, l \in \mathbb{Z}_+$, we have*

- (1) *As a superalgebra, $\mathcal{M}_{0|1}^{k|l} \cong A_q^*$, which is generated by $\xi_1, \xi_2, \dots, \xi_{k+l}$ with parity assignments $[\xi_i] = [i] + \bar{1}$ and defining relations*

$$\begin{aligned} \xi_i^2 &= 0, \quad \text{if } [i] = \bar{0}, \\ \xi_j \xi_i &= (-1)^{([i]+\bar{1})([j]+\bar{1})} q^{-1} \xi_i \xi_j \quad 1 \leq i < j \leq k+l. \end{aligned}$$

- (2) *$A_q^* = \bigoplus_{N \in \mathbb{Z}_+} (A_q^*)_N$, where $(A_q^*)_N$ is the irreducible $U_q(\mathfrak{gl}_{k|l})$ -module with the highest weight $\sum_{i=1}^N \epsilon_i$ if $N < k$, and $\sum_{i=1}^k \epsilon_i + (N-k)\epsilon_{k+1}$ otherwise.*

Proof. This can be proved similarly as Proposition 3.10. \square

The following proposition gives the second formulation of the module superalgebra $\mathcal{M}_{r|s}^{k|l}$, which is a generalisation of [2, Proposition 2.33] to the super case.

Proposition 3.12. *Let $V^{k|l}$ and $V^{r|s}$ be natural modules of $U_q(\mathfrak{gl}_{k|l})$ and $U_q(\mathfrak{gl}_{r|s})$, respectively.*

- (1) *As a superalgebra, $\mathcal{M}_{r|s}^{k|l} \cong S_q(V^{k|l} \otimes V^{r|s})$.*
(2) *As a $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module,*

$$S_q(V^{k|l} \otimes V^{r|s}) \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} L_\lambda^{k|l} \otimes L_\lambda^{r|s}.$$

In particular, $V^{k|l} \otimes V^{r|s}$ is a flat $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module.

Proof. Let $U = V^{k|l} \otimes V^{r|s}$, then it has a basis $\{x_{ia} := v_i \otimes v_a \mid i \in \mathbf{I}_{k|l}, a \in \mathbf{I}_{r|s}\}$, where v_i and v_a are basis elements for $V^{k|l}$ and $V^{r|s}$, respectively. Define the permutation map $P_{23} : U \otimes U \rightarrow U \otimes U$ of two middle factors by

$$P_{23}(v_i \otimes v_j \otimes v_a \otimes v_b) = (-1)^{[j][a]} P_{23}(v_i \otimes v_a \otimes v_j \otimes v_b).$$

Therefore, the R -matrix of $U_q(\mathfrak{gl}_{k|l} \times \mathfrak{gl}_{r|s})$ acting on $U \otimes U$ is $\check{R}_{U,U} = P_{23} \circ (\check{R}_{V^{k|l}, V^{k|l}} \otimes \check{R}_{V^{r|s}, V^{r|s}}) \circ P_{23}$, which implies

$$\Lambda_q^2(U) = P_{23} \left((S_q^2(V^{k|l}) \otimes \Lambda_q^2(V^{r|s})) \oplus (\Lambda_q^2(V^{k|l}) \otimes S_q^2(V^{r|s})) \right).$$

Using bases for $S_q^2(V^{k|l})$ and $\Lambda_q^2(V^{k|l})$ ($S_q^2(V^{r|s})$ and $\Lambda_q^2(V^{r|s})$) given in (3.7) and (3.8), we immediately obtain the following quadratic relations for $S_q(U) = T(U)/\Lambda_q^2(U)$:

$$\begin{aligned} (x_{ia})^2 &= 0, & [i] + [a] &= \bar{1}, \\ x_{ja}x_{ia} &= (-1)^{([i]+[a])([j]+[a])} q_a x_{ia}x_{ja}, & j &> i, \\ x_{ib}x_{ia} &= (-1)^{([i]+[a])([i]+[b])} q_i x_{ia}x_{ib}, & b &> a, \\ x_{ja}x_{ib} &= (-1)^{([i]+[b])([j]+[a])} x_{ib}x_{ja}, & j &> i, a < b, \\ x_{jb}x_{ia} &= (-1)^{([i]+[a])([j]+[b])} x_{ia}x_{jb} \\ &\quad + (-1)^{[i]([j]+[b])+[j][b]} (q - q^{-1}) x_{ib}x_{ja}, & j &> i, b > a. \end{aligned}$$

It is straightforward to verify that the assignment $t_{ia} \mapsto (-1)^{[i][a]} x_{ia}$ preserves defining relations, which becomes the superalgebra isomorphism between $\mathcal{M}_{r|s}^{k|l}$ and $S_q(V^{k|l} \otimes V^{r|s})$. Thus we have

$$\dim_{\mathcal{K}} S_q(V^{k|l} \otimes V^{r|s})_N = \dim_{\mathcal{K}} (\mathcal{M}_{r|s}^{k|l})_N = \sum_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}, |\lambda|=N} \dim_{\mathcal{K}} L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|s},$$

while by Remark 3.6 in the classical case ($q \rightarrow 1$)

$$\dim_{\mathbb{C}} S(\mathbb{C}^{k|l} \otimes \mathbb{C}^{r|s})_N = \sum_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}, |\lambda| \leq N} \dim_{\mathbb{C}} L_{\lambda}^{k|l}|_{q=1} \otimes L_{\lambda}^{r|s}|_{q=1}.$$

Using $\dim_{\mathcal{K}} L_{\lambda}^{k|l} = \dim_{\mathbb{C}} L_{\lambda}^{k|l}|_{q=1}$ for any $k, l \in \mathbb{Z}_+$, we have $\dim_{\mathcal{K}} S_q(V^{k|l} \otimes V^{r|s})_N = \dim_{\mathbb{C}} S(\mathbb{C}^{k|l} \otimes \mathbb{C}^{r|s})_N$. This implies that $V^{k|l} \otimes V^{r|s}$ is a flat $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module. For part (2), it is easy to see that $S_q(V^{k|l} \otimes V^{r|s})$ acquires a $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module structure through the isomorphism given in part (1). Thus the decomposition in part (2) follows from Theorem 3.4. \square

Remark 3.13. We immediately recover the following dualities:

- (1) (skew $(U_q(\mathfrak{gl}_k), U_q(\mathfrak{gl}_s))$ -duality, [16, Theorem 6.16]) As $U_q(\mathfrak{gl}_k) \otimes U_q(\mathfrak{gl}_s)$ -module,

$$\Lambda_q(V^{k|0} \otimes V^{s|0}) \cong S_q(V^{k|0} \otimes V^{0|s}) \cong \bigoplus_{\lambda \in \Lambda_{k|0} \cap \Lambda_{0|s}} L_{\lambda}^{k|0} \otimes L_{\lambda'}^{s|0},$$

where the first isomorphism follows from the defining relations of these two algebras (see [16, Proposition 6.14]), and the second one has used the isomorphism $L_{\lambda}^{0|s} \cong L_{\lambda'}^{s|0}$ as $U_q(\mathfrak{gl}_{s|0})$ -modules (see part (1) of Proposition A.4). Note $U_q(\mathfrak{gl}_s) := U_q(\mathfrak{gl}_{s|0})$.

- (2) $((U_q(\mathfrak{gl}_k), U_q(\mathfrak{gl}_r))$ -duality, [16, Theorem 6.4] and [37]) As $U_q(\mathfrak{gl}_k) \otimes U_q(\mathfrak{gl}_r)$ -module,

$$S_q(V^{k|0} \otimes V^{r|0}) \cong \bigoplus_{\lambda \in \Lambda_{k|0} \cap \Lambda_{r|0}} L_{\lambda}^{k|0} \otimes L_{\lambda}^{r|0}.$$

- (3) $((U_q(\mathfrak{gl}_{k|l}), U_q(\mathfrak{gl}_r))$ -duality, [31, Theorem 2.2]) As $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_r)$ -module

$$S_q(V^{k|l} \otimes V^{r|0}) \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|0}} L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|0}.$$

Proposition 3.14. *As a superalgebra, $\overline{\mathcal{M}}_{r|s}^{k|l} \cong S_q((V^{k|l})^* \otimes (V^{r|s})^*)$. Thus we have the following multiplicity-free decomposition as $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module*

$$S_q((V^{k|l})^* \otimes (V^{r|s})^*) \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} L_{\lambda}^{k|l} \otimes (L_{\lambda}^{r|s})^*.$$

In particular, $(V^{k|l})^ \otimes (V^{r|s})^*$ is a flat $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module.*

Proof. We only give a sketch of the proof, as it is similar to that of Proposition 3.12. Let $v_i^*, i \in \mathbf{I}_{r|s}$, be standard weight basis for $(V^{k|l})^*$. Then $\check{R}_{(V^{r|s})^*, (V^{r|s})^*}$ acts on $(V^{r|s})^* \otimes (V^{r|s})^*$ by

$$(3.10) \quad \check{R}_{(V^{r|s})^*, (V^{r|s})^*}(v_i^* \otimes v_j^*) = \begin{cases} (-1)^{[i][j]} v_j^* \otimes v_i^*, & \text{if } i > j \\ (-1)^{[i]} q_i v_i^* \otimes v_i^*, & \text{if } i = j, \\ (-1)^{[i][j]} v_j^* \otimes v_i^* + (q - q^{-1}) v_i^* \otimes v_j^*, & \text{if } i < j. \end{cases}$$

Let $\bar{U} = (V^{k|l})^* \otimes (V^{r|s})^*$ and P_{23} be graded the permutation of two middle factors in $\bar{U} \otimes \bar{U}$. Then we have

$$\Lambda_q^2(\bar{U}) = P_{23} \left((S_q^2((V^{k|l})^*) \otimes \Lambda_q^2((V^{r|s})^*)) \oplus (\Lambda_q^2((V^{k|l})^*) \otimes S_q^2((V^{r|s})^*)) \right).$$

The bases for $S_q^2((V^{r|s})^*)$ and $\Lambda_q^2((V^{r|s})^*)$ are given respectively by $\{v_i^* \otimes v_i^*, 1 \leq i \leq r, v_i^* \otimes v_j^* + (-1)^{[i][j]} q^{-1} v_j^* \otimes v_i^*, 1 \leq i < j \leq r+s\}$ and $\{v_i^* \otimes v_i^*, r+1 \leq i \leq r+s, v_i^* \otimes v_j^* - (-1)^{[i][j]} q v_j^* \otimes v_i^*, 1 \leq i < j \leq r+s\}$. Now we can determine the quadratic relations for $S_q(\bar{U}) = T(\bar{U})/\Lambda_q^2(\bar{U})$ as follows:

$$\begin{aligned} (x_{ia})^2 &= 0, & [i] + [a] &= \bar{1}, \\ x_{ja} x_{ia} &= (-1)^{([i]+[a])([j]+[a])} q_a^{-1} x_{ia} x_{ja}, & j &> i, \\ x_{ib} x_{ia} &= (-1)^{([i]+[a])([i]+[b])} q_i^{-1} x_{ia} x_{ib}, & b &> a, \\ x_{ja} x_{ib} &= (-1)^{([i]+[b])([j]+[a])} x_{ib} x_{ja}, & j &> i, a < b, \\ x_{jb} x_{ia} &= (-1)^{([i]+[a])([j]+[b])} x_{ia} x_{jb} \\ &\quad - (-1)^{[i]([j]+[b])+[j][b]} (q - q^{-1}) x_{ib} x_{ja}, & j &> i, b > a. \end{aligned}$$

Here $x_{ia} := v_i^* \otimes v_a^*$. The isomorphism between $\bar{\mathcal{M}}_{r|s}^{k|l}$ and $S_q(\bar{U})$ is given by $\bar{t}_{ia} \mapsto (-1)^{[i][a]} x_{ia}$. The multiplicity-free decomposition is also clear from this isomorphism and Remark 3.5. \square

Remark 3.15. We will show in Proposition 4.23 that $V^{k|l} \otimes (V^{r|s})^*$ is a flat $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module.

3.3. Howe duality implies Schur-Weyl duality. We derive the quantum super analogue of Schur-Weyl duality from the quantum super analogue of Howe duality following the methods of [37, Proposition 4.1], and also [12, Proposition 2.4.5.3] for the classical case.

Recall that the Hecke algebra $\mathcal{H}_q(r)$ of the symmetric group is the unital \mathcal{K} -algebra with generators H_1, H_2, \dots, H_{r-1} subject to relations

$$(3.11) \quad \begin{aligned} (H_i - q)(H_i + q^{-1}) &= 0, \quad 1 \leq i \leq r-1, \\ H_i H_j &= H_j H_i, \quad |i - j| \geq 2, \\ H_i H_{i+1} H_i &= H_{i+1} H_i H_{i+1}, \quad 1 \leq i \leq r-2. \end{aligned}$$

It is well known that the Hecke algebra $\mathcal{H}_q(r)$ and the symmetric group algebra $\mathcal{K}\text{Sym}_r$ are isomorphic as associative algebras (cf. [14]). Therefore, the irreducible representations of $\mathcal{H}_q(r)$ are also indexed by partitions $\lambda \vdash r$.

Let $V^{k|l}$ be the natural module for $U_q(\mathfrak{gl}_{k|l})$ with the standard basis $\{v_a\}_{a \in \mathbf{I}_{k|l}}$. Then there is a natural action of $\mathcal{H}_q(r)$ on $(V^{k|l})^{\otimes r}$ defined by

$$H_i \cdot v_{a_1} \otimes v_{a_2} \otimes \dots \otimes v_{a_r} := v_{a_1} \otimes \dots \otimes \check{R}_{V^{k|l}, V^{k|l}}(v_{a_i} \otimes v_{a_{i+1}}) \otimes \dots \otimes v_{a_r},$$

for all $1 \leq i \leq r$. This action is well defined following (3.6) and the quadratic relation (3.9), thus leads to the algebra homomorphism

$$(3.12) \quad \nu_r : \mathcal{H}_q(r) \longrightarrow \text{End}_{U_q(\mathfrak{gl}_{k|l})}((V^{k|l})^{\otimes r}).$$

The quantum super analogue of Schur-Weyl duality (see , e.g. [19, 7]) states that this is actually an epimorphism.

Theorem 3.16. (Schur-Weyl duality) *Let $V^{k|l}$ be the natural representation of $U_q(\mathfrak{gl}_{k|l})$. As $U_q(\mathfrak{gl}_{k|l}) \otimes \mathcal{H}_q(r)$ -module, $(V^{k|l})^{\otimes r}$ has the following multiplicity-free decomposition,*

$$(V^{k|l})^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda_{k|l}, |\lambda|=r} L_\lambda^{k|l} \otimes D_\lambda,$$

where D_λ is the simple left $\mathcal{H}_q(r)$ -module associated to λ . This in particular implies that the algebra homomorphism ν_r defined by (3.12) is surjective.

Proof. It follows from Theorem 3.4 that $\mathcal{M}_{r|0}^{k|l} \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|0}} L_\lambda^{k|l} \otimes L_\lambda^{r|0}$ as $\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \mathcal{R}(U_q(\mathfrak{gl}_r))$ -module, where $L_\lambda^{r|0}$ is the irreducible $U_q(\mathfrak{gl}_r)$ -module. Denote by

$$\mathcal{M}_{r|0}^{k|l}[0] := \{f \in \mathcal{M}_{r|0}^{k|l} \mid \mathcal{R}_{K_a}(f) = qf, \quad \forall 1 \leq a \leq r\}$$

the zero weight space of $\mathcal{M}_{r|0}^{k|l}$ with respect to the \mathcal{R} -action of $U_q(\mathfrak{gl}_r)$. This is isomorphic to $(V^{k|l})^{\otimes r}$ with a basis $\{t_{a_1,1} t_{a_2,2} \dots t_{a_r,r} \mid 1 \leq a_1, a_2, \dots, a_r \leq k+l\}$ by Proposition 3.8. Then we immediately obtain as left $\mathcal{L}(U_q(\mathfrak{gl}_{k|l}))$ -modules

$$\mathcal{M}_{r|0}^{k|l}[0] \cong (V^{k|l})^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda_{k|l}, |\lambda|=r} L_\lambda^{k|l} \otimes L_\lambda^{r|0}[0],$$

where $L_\lambda^{r|0}[0]$ is the zero weight space of $L_\lambda^{r|0}$, and we have used the fact that $L_\lambda^{r|0}[0]$ is nonzero if and only if $|\lambda| = r$. Now it is well known that $L_\lambda^{r|0}[0]$ is actually the irreducible $\mathcal{H}_q(r)$ -module associated to partition λ of size r , see [37, Proposition 4.1] (and [12, Proposition 2.4.5.3] for the classical case). This proves the multiplicity-free decomposition. The second statement is clear. \square

4. THE FFT OF INVARIANT THEORY FOR $U_q(\mathfrak{gl}_{m|n})$

4.1. Invariants for $U_q(\mathfrak{gl}_{m|n})$. Fix non-negative integers k, l, r, s . In this section, we shall work with the following $U_q(\mathfrak{gl}_{m|n})$ -module superalgebra.

Definition 4.1. Let $\mathcal{P}_{r|s}^{k|l} := \mathcal{M}_{m|n}^{k|l} \otimes_{\mathfrak{R}} \overline{\mathcal{M}}_{m|n}^{r|s}$, the braided tensor algebra of $\mathcal{M}_{m|n}^{k|l}$ and $\overline{\mathcal{M}}_{m|n}^{r|s}$. Then it is a $U_q(\mathfrak{gl}_{m|n})$ -module superalgebra with respect to the action $\mathcal{R}(U_q(\mathfrak{gl}_{m|n})) \otimes \mathcal{R}(U_q(\mathfrak{gl}_{m|n}))$, defined for any $f \otimes g \in \mathcal{P}_{r|s}^{k|l}$ and $x \in U_q(\mathfrak{gl}_{m|n})$ by

$$x(f \otimes g) = \sum_{(x)} (-1)^{[x_{(2)}][f]} \mathcal{R}_{x_{(1)}}(f) \otimes \mathcal{R}_{x_{(2)}}(g).$$

Remark 4.2. By Proposition 3.12 and Proposition 3.14, $\mathcal{P}_{r|s}^{k|l}$ is a flat deformation of the supersymmetric algebra $S(\mathbb{C}^{k|l} \otimes \mathbb{C}^{m|n} \oplus (\mathbb{C}^{r|s})^* \otimes (\mathbb{C}^{m|n})^*)$, which is isomorphic to $S(\mathbb{C}^{k|l} \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{r|s} \otimes (\mathbb{C}^{m|n})^*)$ as superalgebra.

The superalgebra algebraic structure of $\mathcal{P}_{r|s}^{k|l}$ can be described as follows.

Lemma 4.3. *Let $T_{ai} = t_{ai} \otimes 1$ and $\overline{T}_{bj} = 1 \otimes \bar{t}_{bj}$ with $a \in \mathbf{I}_{k|l}$, $b \in \mathbf{I}_{r|s}$ and $i, j \in \mathbf{I}_{m|n}$. Then $\mathcal{P}_{r|s}^{k|l}$ is generated by the elements T_{ai} and \overline{T}_{bj} subject to the following relations for all $a \in \mathbf{I}_{k|l}$, $b \in \mathbf{I}_{r|s}$ and $i, j \in \mathbf{I}_{m|n}$:*

$$\begin{aligned} (4.1) \quad & T_{ai} \text{ satisfy the relevant relations for } t_{ai} \text{ in (2.9),} \\ & \overline{T}_{bi} \text{ satisfy the relevant relations for } \bar{t}_{bi} \text{ in (2.17),} \\ & \overline{T}_{bj} T_{ai} = (-1)^{([a]+[i])([b]+[j])} T_{ai} \overline{T}_{bj}, \quad i \neq j \\ & \overline{T}_{bi} T_{ai} = (-1)^{([a]+[i])([b]+[i])} q_i^{-1} T_{ai} \overline{T}_{bi} - \sum_{j>i} (-1)^{[b]([a]+[i])+[a][j]} (q - q^{-1}) T_{aj} \overline{T}_{bj}. \end{aligned}$$

Proof. All the statements in the lemma are obvious except the last two relations, which require proof. Note that they are equivalent to

$$(4.2) \quad \bar{T}_{bj}T_{ai} = \sum_{a', b' \in \mathbf{I}_{m|n}} (-1)^{([a]+[i])([b]+[b'])+([i]+[j]+[b'])([j]+[b'])} (R^{-1})_{b'i}^{j a'} T_{aa'} \bar{T}_{bb'}.$$

Let $\mathfrak{R} = \sum_h \alpha_h \otimes \beta_h \in U_q(\mathfrak{gl}_{m|n}) \otimes U_q(\mathfrak{gl}_{m|n})$ be the universal \mathfrak{R} matrix. It follows from Proposition 2.3 that

$$(1 \otimes \bar{t}_{bj})(t_{ai} \otimes 1) = P\mathfrak{R}(\bar{t}_{bj} \otimes t_{ai}) = \sum_h (-1)^{[\alpha_h][\beta_h]+([\alpha_h]+[j]+[b])([i]+[a])} \mathcal{R}_{\beta_h}(t_{ai}) \otimes \mathcal{R}_{\alpha_h}(\bar{t}_{bj}),$$

where we have the \mathcal{R} -actions

$$\begin{aligned} \mathcal{R}_{\beta_h}(t_{ai}) &= \sum_{a' \in \mathbf{I}_{m|n}} (-1)^{([a]+[a'])([i]+[a'])+([\beta_h]+[i]+[a])([\beta_h])} t_{aa'} \langle t_{a'i}, \beta_h \rangle, \\ \mathcal{R}_{\alpha_h}(\bar{t}_{bj}) &= \sum_{b' \in \mathbf{I}_{m|n}} (-1)^{([b]+[b'])([b']+[j])+([\alpha_h]+[j]+[b])([\alpha_h]+[j])([j]+[b'])} \bar{t}_{bb'} \langle t_{jb'}, S(\alpha_h) \rangle. \end{aligned}$$

Recall $\mathfrak{R}^{-1} = \sum_h S(\alpha_h) \otimes \beta_h$ and $R^{-1} = (\pi \otimes \pi)\mathfrak{R}^{-1} = \sum_{a,b,c,d \in \mathbf{I}_{m|n}} e_{ab} \otimes e_{cd} (R^{-1})_{bd}^{ac}$. Note that $\sum_h \langle t_{jb'}, S(\alpha_h) \rangle \langle t_{a'i}, \beta_h \rangle = (R^{-1})_{b'i}^{j a'}$ with the following nonzero entries of R^{-1}

$$(4.3) \quad (R^{-1})_{ab}^{a b} = 1, \quad a \neq b, \quad (R^{-1})_{aa}^{a a} = q_a^{-1}, \quad (R^{-1})_{ba}^{a b} = -(q_b - q_b^{-1}), \quad a < b.$$

Using above equations together, we obtain (4.2) as required. \square

We want to study the subalgebra of $U_q(\mathfrak{gl}_{m|n})$ -invariants in $\mathcal{P}_{r|s}^{k|l}$ (cf. Lemma 2.5). Let

$$(4.4) \quad \mathcal{X}_{r|s}^{k|l} := (\mathcal{P}_{r|s}^{k|l})^{U_q(\mathfrak{gl}_{m|n})} = \{f \in \mathcal{P}_{r|s}^{k|l} \mid x(f) = \epsilon(x)f, \forall x \in U_q(\mathfrak{gl}_{m|n})\}.$$

We define the following elements in $\mathcal{P}_{r|s}^{k|l}$ by

$$(4.5) \quad X_{ab} := \sum_{i \in \mathbf{I}_{m|n}} (-1)^{[a]([b]+[i])} t_{ai} \otimes \bar{t}_{bi} = \sum_{i \in \mathbf{I}_{m|n}} (-1)^{[a]([b]+[i])} T_{ai} \bar{T}_{bi}$$

for all $a \in \mathbf{I}_{k|l}$ and $b \in \mathbf{I}_{r|s}$. The \mathbb{Z}_2 -gradation is given by $[X_{ab}] = [a] + [b]$.

Lemma 4.4. *The elements X_{ab} belong to $\mathcal{X}_{r|s}^{k|l}$ for all $a \in \mathbf{I}_{k|l}$ and $b \in \mathbf{I}_{r|s}$.*

Proof. For all $x \in U_q(\mathfrak{gl}_{m|n})$, we have

$$(4.6) \quad xX_{ab} = \sum_{(x)} \sum_{i \in \mathbf{I}_{m|n}} (-1)^{[x_{(2)}]([a]+[i])+[a]([b]+[i])} \mathcal{R}_{x_{(1)}}(t_{ai}) \otimes \mathcal{R}_{x_{(2)}}(\bar{t}_{bi}).$$

Note that

$$(4.7) \quad \begin{aligned} \mathcal{R}_{x_{(1)}}(t_{ai}) &= \sum_{c \in \mathbf{I}_{m|n}} (-1)^{[x_{(1)}]([a]+[i]+[x_{(1)}])+([a]+[c])([c]+[i])} t_{ac} \langle t_{ci}, x_{(1)} \rangle \\ \mathcal{R}_{x_{(2)}}(\bar{t}_{bi}) &= \sum_{d \in \mathbf{I}_{m|n}} (-1)^{[x_{(2)}]([b]+[i]+[x_{(2)}])+([b]+[d])([d]+[i])} \bar{t}_{bd} \langle \bar{t}_{di}, x_{(2)} \rangle, \end{aligned}$$

where $\langle t_{ci}, x_{(1)} \rangle = \pi(x_{(1)})_{ci}$ and $\langle \bar{t}_{di}, x_{(2)} \rangle = (-1)^{[i]([d]+[i])} \pi(S(x_{(2)}))_{id}$. Observe that

$$\pi(x_{(1)})_{ci} \neq 0 \Leftrightarrow [x_{(1)}] = [c] + [i], \quad \pi(S(x_{(2)}))_{id} \neq 0 \Leftrightarrow [x_{(2)}] = [i] + [d].$$

Combing (4.6) and (4.7), we obtain

$$\begin{aligned} xX_{ab} &= \sum_{(x)} \sum_{i,c,d \in \mathbf{I}_{m|n}} (-1)^{[a]([b]+[d])} \pi(x_{(1)})_{ci} \pi(S(x_{(2)}))_{id} T_{ac} \bar{T}_{bd} \\ &= \sum_{c,d \in \mathbf{I}_{m|n}} (-1)^{[a]([b]+[d])} \pi(\epsilon(x))_{cd} T_{ac} \bar{T}_{bd} \\ &= \epsilon(x) X_{ab}, \end{aligned}$$

where we have used the identity $\sum_{(x)} x_{(1)} S(x_{(2)}) = \epsilon(x)1$. \square

Lemma 4.5. (1) The following relations hold in $\mathcal{P}_{r|s}^{k|l}$:

$$\begin{aligned}
X_{ac}T_{bi} &= (-1)^{([b]+[i])([a]+[c])}T_{bi}X_{ac}, & a > b, \\
X_{ac}T_{ai} &= (-1)^{([a]+[i])([a]+[c])}q_a^{-1}T_{ai}X_{ac}, \\
T_{ai}X_{bc} - (-1)^{([a]+[i])([b]+[c])}X_{bc}T_{ai} &= (-1)^{[c]([a]+[b])+[a][b]}(q - q^{-1})T_{bi}X_{ac}, & a > b, \\
X_{ab}\overline{T}_{bi} &= (-1)^{([a]+[b])([b]+[i])}q_b\overline{T}_{bi}X_{ai}, \\
X_{ab}\overline{T}_{ci} &= (-1)^{([a]+[b])([c]+[i])}\overline{T}_{ci}X_{ab}, & b > c, \\
X_{ac}\overline{T}_{bi} - (-1)^{([b]+[i])([a]+[c])}\overline{T}_{bi}X_{ac} &= (-1)^{[i]([a]+[c])+[a][b]}(q - q^{-1})\overline{T}_{ci}X_{ab}, & b > c.
\end{aligned}$$

(2) The elements $X_{ab} \in \mathcal{X}_{r|s}^{k|l}$, $a \in \mathbf{I}_{k|l}$, $b \in \mathbf{I}_{r|s}$ satisfy the following relations

$$\begin{aligned}
(X_{ab})^2 &= 0, & [a] + [b] &= \bar{1}, \\
X_{ac}X_{bc} &= (-1)^{([a]+[c])([b]+[c])}q_cX_{bc}X_{ac}, & a > b, \\
X_{ab}X_{ac} &= (-1)^{([a]+[b])([a]+[c])}q_a^{-1}X_{ac}X_{ab}, & b > c, \\
X_{ac}X_{bd} &= (-1)^{([a]+[c])([b]+[d])}X_{bd}X_{ac}, & a > b, c > d, \\
X_{ac}X_{bd} &= (-1)^{([a]+[c])([b]+[d])}X_{bd}X_{ac} \\
&\quad + (-1)^{[a]([b]+[d])+[b][d]}(q - q^{-1})X_{bc}X_{ad}, & a > b, c < d.
\end{aligned} \tag{4.8}$$

Proof. The proof for part (1) is rather involved as there are many cases to consider, even though all are quite similar. We illustrate the proof by using the third relation in part (1) as an example. Using relations (4.1), we obtain

$$\begin{aligned}
T_{ai}X_{bc} &= \sum_{j \in \mathbf{I}_{m|n}} (-1)^{[b]([c]+[j])}T_{ai}T_{bj}\overline{T}_{cj} = S_{i>j} + S_{i<j} + S_{i=j}, \text{ with} \\
S_{j<i} &= \sum_{j<i} (-1)^{[b]([c]+[j])} \left((-1)^{([a]+[i])([b]+[j])}T_{bj}T_{ai}\overline{T}_{cj} \right. \\
&\quad \left. + (-1)^{[a]([b]+[j])+[b][j]}(q - q^{-1})T_{bi}T_{aj}\overline{T}_{cj} \right), \\
S_{j>i} &= \sum_{j>i} (-1)^{[b]([c]+[j])+([a]+[i])([b]+[j])}T_{bj}T_{ai}\overline{T}_{cj}, \\
S_{j=i} &= (-1)^{[b]([c]+[i])+([a]+[i])([b]+[i])}q_iT_{bi}T_{ai}\overline{T}_{ci}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
X_{bc}T_{ai} &= \sum_{j \in \mathbf{I}_{m|n}} (-1)^{[b]([c]+[j])}T_{bj}\overline{T}_{cj}T_{ai} = S'_{j=i} + S'_{j \neq i} \text{ with} \\
S'_{j=i} &= (-1)^{[b]([c]+[i])} \left((-1)^{([a]+[i])([c]+[i])}q_i^{-1}T_{bi}T_{ai}\overline{T}_{ci} \right. \\
&\quad \left. - \sum_{k>i} (-1)^{[c]([a]+[i])+[a][k]}(q - q^{-1})T_{bi}T_{ak}\overline{T}_{ck} \right), \\
S'_{j \neq i} &= \sum_{j \neq i} (-1)^{[b]([c]+[j])+([a]+[i])([c]+[j])}T_{bj}T_{ai}\overline{T}_{cj}.
\end{aligned}$$

Straightforward calculation shows that

$$T_{ai}X_{bc} - (-1)^{([a]+[i])([b]+[c])}X_{bc}T_{ai} = (-1)^{[c]([a]+[b])+[a][b]}(q - q^{-1})T_{bi}X_{ac}, \quad a > b.$$

Now we turn to part (2), which is an easy consequence of part (1). For instance, if $[a] + [b] = \bar{1}$, then

$$\begin{aligned} (X_{ab})^2 &= \sum_{i \in \mathbf{I}_{m|n}} (-1)^{[a]([b]+[i])} X_{ab} T_{ai} \bar{T}_{bi} \\ &= \sum_{i \in \mathbf{I}_{m|n}} (-1)^{[a]([b]+[i]) + ([a]+[b])([a]+[i])} q_a^{-1} T_{ai} X_{ab} \bar{T}_{bi} \\ &= \sum_{i \in \mathbf{I}_{m|n}} (-1)^{[a]([b]+[i]) + ([a]+[b])} q_a^{-1} q_b T_{ai} \bar{T}_{ai} X_{ab} \\ &= -q_a^{-1} q_b (X_{ab})^2. \end{aligned}$$

Hence $(X_{ab})^2 = 0$ in this case. The remaining relations can be proved in a similar way. \square

The following is one of the main results of this paper.

Theorem 4.6. (FFT for $U_q(\mathfrak{gl}_{m|n})$) *The invariant subalgebra $\mathcal{X}_{r|s}^{k|l}$ is generated by the elements X_{ab} with $a \in \mathbf{I}_{k|l}$ and $b \in \mathbf{I}_{r|s}$.*

Remark 4.7. The special case with $n = 0$ of the theorem is the FFT of invariant theory for the quantum general linear group [16]. One can also recover from the theorem the FFT of invariant theory for the general linear Lie supergroup [15, 27, 28] in the $q \rightarrow 1$ limit. These points will be discussed in Section 6.

4.2. Proof of FFT. The proof of Theorem 4.6 will be divided into two cases based on the following lemma.

Lemma 4.8. *The invariant subalgebra $\mathcal{X}_{r|s}^{k|l}$ admits the multiplicity-free decomposition*

$$(4.9) \quad \mathcal{X}_{r|s}^{k|l} \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s} \cap \Lambda_{m|n}} L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|s}$$

as $\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{r|s}))$ -module.

Proof. By Definition 4.1 and Theorem 3.4, we obtain that as $\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \mathcal{R}(U_q(\mathfrak{gl}_{m|n})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{r|s})) \otimes \mathcal{R}(U_q(\mathfrak{gl}_{m|n}))$ -module,

$$\mathcal{P}_{r|s}^{k|l} \cong \bigoplus_{\substack{\lambda \in \Lambda_{k|l} \cap \Lambda_{m|n}, \\ \mu \in \Lambda_{r|s} \cap \Lambda_{m|n}}} L_{\lambda}^{k|l} \otimes L_{\lambda}^{m|n} \otimes L_{\mu}^{r|s} \otimes (L_{\mu}^{m|n})^*.$$

It follows that

$$\begin{aligned} \mathcal{X}_{r|s}^{k|l} &\cong \bigoplus_{\substack{\lambda \in \Lambda_{k|l} \cap \Lambda_{m|n}, \\ \mu \in \Lambda_{r|s} \cap \Lambda_{m|n}}} L_{\lambda}^{k|l} \otimes L_{\mu}^{r|s} \otimes (L_{\lambda}^{m|n} \otimes (L_{\mu}^{m|n})^*)^{U_q(\mathfrak{gl}_{m|n})} \\ &\cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s} \cap \Lambda_{m|n}} L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|s}, \end{aligned}$$

where we have used Schur's Lemma that $(L_{\lambda}^{m|n} \otimes (L_{\mu}^{m|n})^*)^{U_q(\mathfrak{gl}_{m|n})}$ has a unique invariant (up to scalar multiples) if and only if $\lambda = \mu$. \square

Inspired by the approach to FFT of invariant theory in [16], we are in a position to deal with the following two cases respectively:

- $m \geq \max\{k, r\}$ and $n \geq \max\{l, s\}$;
- $m < \max\{k, r\}$ or $n < \max\{l, s\}$.

4.2.1. *The case $m \geq \max\{k, r\}$ and $n \geq \max\{l, s\}$.* In this case, one immediately obtains

$$(4.10) \quad \Lambda_{k|l} \cap \Lambda_{r|s} \cap \Lambda_{m|n} = \Lambda_{k|l} \cap \Lambda_{r|s}.$$

($m \geq \max\{k, r\}$ and $n \geq \max\{l, s\}$ is only a sufficient condition for this equation.) Viewing $\mathcal{M}_{r|s}^{k|l}$ as a subalgebra of $\mathcal{M}_{m|n}$, we have $\Delta(\mathcal{M}_{r|s}^{k|l}) \subset \mathcal{M}_{m|n}^{k|l} \otimes \mathcal{M}_{r|s}^{m|n}$ under the co-multiplication on $\mathcal{M}_{m|n}$. Let $\tilde{\Delta} := (\text{id} \otimes S)\Delta$ be the \mathcal{K} -linear map

$$\tilde{\Delta} : \mathcal{M}_{r|s}^{k|l} \longrightarrow \mathcal{P}_{r|s}^{k|l} = \mathcal{M}_{m|n}^{k|l} \otimes_{\mathfrak{R}} \overline{\mathcal{M}}_{m|n}^{r|s},$$

where S is the antipode of $\mathcal{M}_{m|n}$ restricted on $\mathcal{M}_{r|s}^{m|n}$.

Lemma 4.9. *$\tilde{\Delta}$ is injective.*

Proof. We have

$$(\epsilon \otimes \text{id})\tilde{\Delta}(f) = \sum_{(f)} 1 \otimes \epsilon(f_{(1)})S(f_{(2)}) = S(f), \quad \forall f \in \mathcal{M}_{r|s}^{k|l}.$$

Thus, $\tilde{\Delta}(f) = 0$ if and only if $f = 0$, as S is invertible. \square

Lemma 4.10. *Assume that $m \geq \max\{k, r\}$ and $n \geq \max\{l, s\}$, then $\tilde{\Delta}(\mathcal{M}_{r|s}^{k|l}) = \mathcal{X}_{r|s}^{k|l}$. In particular, $\tilde{\Delta}(t_{ab}) = X_{ab} \in \mathcal{X}_{r|s}^{k|l}, \forall a \in \mathbf{I}_{k|l}, b \in \mathbf{I}_{r|s}$.*

Proof. By (4.10), Lemma 4.8 and Theorem 3.4, we have

$$\dim_{\mathcal{K}}(\mathcal{X}_{r|s}^{k|l})_N = \sum_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}, |\lambda|=N} \dim_{\mathcal{K}} L_{\lambda}^{k|l} \times \dim_{\mathcal{K}} L_{\lambda}^{r|s} = \dim_{\mathcal{K}}(\mathcal{M}_{r|s}^{k|l})_N,$$

for any homogeneous components of degree N . We only need to show that $\tilde{\Delta}(\mathcal{M}_{r|s}^{k|l}) \subseteq \mathcal{X}_{r|s}^{k|l}$, since the identity follows from the injectivity of $\tilde{\Delta}$. Now for any $x \in U_q(\mathfrak{gl}_{m|n})$ and $f \in \mathcal{M}_{r|s}^{k|l}$, we have

$$\begin{aligned} x\tilde{\Delta}(f) &= \sum_{(x), (f)} (-1)^{[x_{(2)}][f_{(1)}]} \mathcal{R}_{x_{(1)}}(f_{(1)}) \otimes \mathcal{R}_{x_{(2)}}(S(f_{(2)})), \\ &= \sum_{(x), (f)} (-1)^{[x_{(2)}]([f_{(1)}] + [f_{(2)}])} X_1 \otimes X_2, \end{aligned}$$

with

$$\begin{aligned} X_1 &= (-1)^{([f_{(1)}] + [f_{(2)}] + [x_{(1)}])[x_{(1)}]} f_{(1)} \langle f_{(2)}, x_{(1)} \rangle, \\ X_2 &= (-1)^{([f_{(3)}] + [f_{(4)}] + [x_{(2)}])[x_{(2)}] + [f_{(3)}][f_{(4)}]} S(f_{(4)}) \langle S(f_{(3)}), x_{(2)} \rangle. \end{aligned}$$

Note that we have used the identity $\Delta(S(f)) = \sum_{(f)} (-1)^{[f_{(1)}][f_{(2)}]} S(f_{(2)}) \otimes S(f_{(1)})$. Thus, we obtain

$$x\tilde{\Delta}(f) = \sum_{(f)} (-1)^{[f_{(1)}][x]} \epsilon(x) f_{(1)} \otimes S(f_{(2)}) = \epsilon(x) \tilde{\Delta}(f),$$

completing our proof. The second claim is easy to see by the antipode S in (2.16). \square

To complete the proof of Theorem 4.6, we need the following key lemma, which is an extension of [16, Lemma 6.13] to the super setting.

Lemma 4.11. *Let μ be the multiplication in $\mathcal{M}_{r|s}^{k|l}$. Suppose that $m \geq \max\{k, r\}$ and $n \geq \max\{l, s\}$,*

(1) *There exists the $\mathbb{Z}_+ \times \mathbb{Z}_+$ -graded vector space bijection:*

$$\begin{aligned} \varpi : \mathcal{M}_{r|s}^{k|l} \otimes \mathcal{M}_{r|s}^{k|l} &\longrightarrow \mathcal{M}_{r|s}^{k|l} \otimes \mathcal{M}_{r|s}^{k|l}, \\ f \otimes g &\mapsto \sum_{(f), (g)} (-1)^{[f_{(2)}][g_{(1)}]} f_{(1)} \otimes g_{(1)} \langle f_{(2)} \otimes g_{(2)}, \mathfrak{R}^{-1} \rangle. \end{aligned}$$

In particular, for any degree N ($N \in \mathbb{Z}_+$) graded component of $\mathcal{M}_{r|s}^{k|l}$,

$$(\mathcal{M}_{r|s}^{k|l})_N = \mu \circ \varpi((\mathcal{M}_{r|s}^{k|l})_{N-1} \otimes (\mathcal{M}_{r|s}^{k|l})_1).$$

(2) Let $f, g \in \mathcal{M}_{r|s}^{k|l}$, we have the multiplication formula in $\mathcal{X}_{r|s}^{k|l}$ given by

$$\tilde{\Delta}(f)\tilde{\Delta}(g) = \tilde{\Delta} \circ \mu \circ \varpi(f \otimes g).$$

Proof. For part (1), we only need to show that $\varpi(f \otimes g)$ is contained in $\mathcal{M}_{r|s}^{k|l} \otimes \mathcal{M}_{r|s}^{k|l}$, since \mathfrak{R} is invertible. Viewing $\mathcal{M}_{r|s}^{k|l}$ as a subalgebra of $\mathcal{M}_{m|n}$ obtained by the truncation (3.2) and recalling the labelling set $\hat{\mathbf{I}}_{k|l}$ therein, we have for any $a, c \in \hat{\mathbf{I}}_{k|l}$ and $b, d \in \hat{\mathbf{I}}_{r|s}$,

$$\varpi(t_{ab} \otimes t_{cd}) = \sum_{a', b'} \text{sgn } t_{aa'} \otimes t_{cc'} \langle t_{a'b} \otimes t_{c'd}, \mathfrak{R}^{-1} \rangle,$$

with $\text{sgn} = (-1)^{([a]+[a'])([a']+[b])+([c]+[c'])([c']+[d])+([c]+[c'])([a']+[b])}$. Using the matrix form of R^{-1} in (4.3), we obtain that $\langle t_{a'b} \otimes t_{c'd}, \mathfrak{R}^{-1} \rangle = 0$ unless $a', c' \in \hat{\mathbf{I}}_{r|s}$. In general, for any ordered monomials $f = t_{a_1 b_1} \cdots t_{a_M b_M}$ and $g = t_{c_1 d_1} \cdots t_{c_N d_N}$ with $b_i, d_i \in \hat{\mathbf{I}}_{r|s}$, we have $\langle f \otimes g, \mathfrak{R}^{-1} \rangle = 0$ unless $a_i, c_i \in \hat{\mathbf{I}}_{r|s}$ by the defining property of \mathfrak{R} (2.6).

To prove part (2), we need the following technical lemma.

Lemma 4.12. *Maintaining above notation, we have*

$$\begin{aligned} \sum_{(f), (g)} (-1)^{[g(1)][f(2)]} f_{(2)} g_{(2)} \langle f_{(1)} \otimes g_{(1)}, \mathfrak{R} \rangle &= \sum_{(f), (g)} (-1)^{([f(1)]+[f(2)])[g(1)]} g_{(1)} f_{(1)} \langle f_{(2)} \otimes g_{(2)}, \mathfrak{R} \rangle; \\ \sum_{(f), (g)} (-1)^{[f(2)][g(1)]} f_{(1)} g_{(1)} \langle f_{(2)} \otimes g_{(2)}, \mathfrak{R}^{-1} \rangle &= \sum_{(f), (g)} (-1)^{([g(1)]+[g(2)])[f(2)]} g_{(2)} f_{(2)} \langle f_{(1)} \otimes g_{(1)}, \mathfrak{R}^{-1} \rangle. \end{aligned}$$

We postpone the proof of Lemma 4.12 to the end. Let $\mathfrak{R} = \sum_h \alpha_h \otimes \beta_h$ and hence $\mathfrak{R}^{-1} = \sum_h S(\alpha_h) \otimes \beta_h$. Now for any $f, g \in \mathcal{M}_{r|s}^{k|l}$,

$$\begin{aligned} \tilde{\Delta}(f)\tilde{\Delta}(g) &= \sum_{(f), (g), h} (-1)^{([\alpha_h]+[f(2)])[g(1)]+[\alpha_h][\beta_h]} f_{(1)} \mathcal{R}_{\beta_h}(g_{(1)}) \otimes \mathcal{R}_{\alpha_h}(S(f_{(2)})) S(g_{(2)}) \\ &= \sum_{(f), (g), h} (-1)^{([\alpha_h]+[f(2)]+[f(3)])([g(1)]+[g(2)]+[\alpha_h][\beta_h])} X_1 \otimes X_2 \end{aligned}$$

with

$$\begin{aligned} X_1 &= (-1)^{([\beta_h]+[g(1)]+[g(2)])([\beta_h])} f_{(1)} g_{(1)} \langle g_{(2)}, \beta_h \rangle, \\ X_2 &= (-1)^{([\alpha_h]+[f(2)]+[f(3)])([\alpha_h]+[f(2)][f(3)]+[f(3)][g(3)])} S(g_{(3)} f_{(3)}) \langle S(f_{(2)}), \alpha_h \rangle. \end{aligned}$$

The sum is over $(f), (g), h$ such that $[\alpha_h] = [f(2)]$, $[\beta_h] = [g(2)]$ and $[f(2)] + [g(2)] = \bar{0}$ since

$$\langle S(f_{(2)}), \alpha_h \rangle \langle g_{(2)}, \beta_h \rangle = (-1)^{[f(2)][g(2)]} \langle f_{(2)} \otimes g_{(2)}, \mathfrak{R}^{-1} \rangle.$$

Using the second identity in Lemma 4.12, we obtain

$$\begin{aligned} \tilde{\Delta}(f)\tilde{\Delta}(g) &= \sum_{(f), (g)} (-1)^{[f(3)]([g(1)]+[g(2)])+[g(1)][g(2)]+[f(3)][g(3)]} f_{(1)} g_{(1)} S(g_{(3)} f_{(3)}) \langle f_{(2)} \otimes g_{(2)}, \mathfrak{R}^{-1} \rangle \\ &= \sum_{(f), (g)} (-1)^{[f(3)]([g(1)]+[g(2)])+[g(1)][g(2)]} f_{(1)} g_{(1)} S(f_{(2)} g_{(2)}) \langle f_{(3)} \otimes g_{(3)}, \mathfrak{R}^{-1} \rangle \\ &= \sum_{(f), (g)} (-1)^{[f(2)][g(1)]} \tilde{\Delta}(f_{(1)} g_{(1)}) \langle f_{(2)} \otimes g_{(2)}, \mathfrak{R}^{-1} \rangle \\ &= \tilde{\Delta} \circ \mu \circ \varpi(f \otimes g). \end{aligned}$$

This completes the proof of Lemma 4.11. \square

Remark 4.13. Part (2) of Lemma 4.11 implies that the \mathcal{K} -linear map $\tilde{\Delta}$ is *not* a superalgebra homomorphism.

Proof of Lemma 4.12. For our purpose, we only prove the second identity, while the first one can be proved in the same way. Let $\mathfrak{R}^{-1} = \sum_h S(\alpha_h) \otimes \beta_h$, then from (2.5) we have $\Delta(x)\mathfrak{R}^{-1} = \mathfrak{R}^{-1}\Delta'(x)$, which implies that for all $x \in U_q(\mathfrak{gl}_{m|n})$

$$\begin{aligned} & \sum_{(x),h} (-1)^{[\alpha_h][x_{(2)}]} x_{(1)} S(\alpha_h) \otimes x_{(2)} \beta_h \otimes x_{(3)} \\ &= \sum_{(x),h} (-1)^{[x_{(2)}]([\beta_h] + [x_{(1)}])} S(\alpha_h) x_{(2)} \otimes \beta_h x_{(1)} \otimes x_{(3)}. \end{aligned}$$

Now applying $f \otimes g \otimes 1$ to both sides of the above equation, we obtain the left hand side

$$\text{LHS} = \sum_{(x),(f),(g),h} \text{sgn} \langle f_{(1)}, x_{(1)} \rangle \langle f_{(2)}, S(\alpha_h) \rangle \langle g_{(1)}, x_{(2)} \rangle \langle g_{(2)}, \beta_h \rangle x_{(3)},$$

with $\text{sgn} = (-1)^{[\alpha_h][x_{(2)}] + [f_{(2)}][x_{(1)}] + [g_{(2)}][x_{(2)}] + ([g_{(1)}] + [g_{(2)}])([x_{(1)}] + [\alpha_h])}$. The sum is over $(x), (f), (g), h$ such that $[x_{(1)}] = [f_{(1)}]$, $[\alpha_h] = [f_{(2)}]$, $[x_{(2)}] = [g_{(1)}]$, $[\beta_h] = [g_{(2)}]$ and $[f_{(2)}] + [g_{(2)}] = 0$, yielding

$$\text{LHS} = \sum_{(x),(f),(g)} (-1)^{[g_{(1)}][f_{(2)}]} \langle f_{(1)} g_{(1)}, x_{(1)} \rangle x_{(2)} \langle f_{(2)} \otimes g_{(2)}, \mathfrak{R}^{-1} \rangle.$$

The right hand side can be simplified in a similar way, and hence we obtain the equality

$$\begin{aligned} & \sum_{(f),(g),(x)} (-1)^{[f_{(2)}][g_{(1)}]} \langle f_{(1)} g_{(1)}, x_{(1)} \rangle x_{(2)} \langle f_{(2)} \otimes g_{(2)}, \mathfrak{R}^{-1} \rangle \\ &= \sum_{(f),(g),(x)} (-1)^{([g_{(1)}] + [g_{(2)}])[f_{(2)}]} \langle g_{(2)} f_{(2)}, x_{(1)} \rangle x_{(2)} \langle f_{(1)} \otimes g_{(1)}, \mathfrak{R}^{-1} \rangle. \end{aligned}$$

We view $f_{(1)} g_{(1)}$ and $g_{(2)} f_{(2)}$ as linear functions on $U_q(\mathfrak{gl}_{m|n})$. Applying the co-unit ϵ to both sides on $x_{(2)}$ and using $\sum_{(x)} x_{(1)} \epsilon(x_{(2)}) = x$, we obtain our assertion in the lemma. \square

Now we prove Theorem 4.6 in the case that $m \geq \max\{k, r\}$ and $n \geq \max\{l, s\}$.

Proof of Theorem 4.6. By Lemma 4.10, $\tilde{\Delta}(\mathcal{M}_{r|s}^{k|l}) = \mathcal{X}_{r|s}^{k|l}$ and we have seen $\tilde{\Delta}(t_{ab}) = X_{ab}$. We use induction on the degree N of $(\mathcal{M}_{r|s}^{k|l})_N$. As $(\mathcal{M}_{r|s}^{k|l})_N = \mu \circ \varpi((\mathcal{M}_{r|s}^{k|l})_{N-1} \otimes (\mathcal{M}_{r|s}^{k|l})_1)$ by part (1) of Lemma 4.11, the theorem follows directly from part (2) of Lemma 4.11 and induction. \square

4.2.2. The case $m < \max\{k, r\}$ or $n < \max\{l, s\}$. Let $u = \min\{k, r, m\}$ and $v = \min\{l, s, n\}$. Note that $\mathcal{M}_{m|n}^{u|v}$ (resp. $\overline{\mathcal{M}}_{m|n}^{u|v}$) can be embedded into $\mathcal{M}_{m|n}^{k|l}$ (resp. $\overline{\mathcal{M}}_{m|n}^{r|s}$) by using the truncation procedure as in (3.2). Explicitly, we define the commutative subalgebras respectively by

$$\begin{aligned} \Upsilon_{u|v}^{k|l} &:= \langle K_a \mid a \in \{1, 2, \dots, k-u\} \cup \{k+v+1, \dots, k+l\} \rangle \subseteq U_q(\mathfrak{gl}_{k|l}), \\ \Upsilon_{u|v}^{r|s} &:= \langle K_a \mid a \in \{1, 2, \dots, r-u\} \cup \{r+v+1, \dots, r+s\} \rangle \subseteq U_q(\mathfrak{gl}_{r|s}). \end{aligned}$$

Then we obtain

$$(4.11) \quad \mathcal{M}_{m|n}^{u|v} = (\mathcal{M}_{m|n}^{k|l})^{\mathcal{L}(\Upsilon_{u|v}^{k|l})}, \quad \overline{\mathcal{M}}_{m|n}^{u|v} = (\overline{\mathcal{M}}_{m|n}^{r|s})^{\tilde{\mathcal{L}}(\Upsilon_{u|v}^{r|s})}.$$

Using the fact that $\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{r|s}))$ graded-commutes with the $U_q(\mathfrak{gl}_{m|n})$ -action on $\mathcal{P}_{r|s}^{k|l}$, we define

$$(4.12) \quad \mathcal{X}_{u|v} := (\mathcal{X}_{r|s}^{k|l})^{\mathcal{L}(\Upsilon_{u|v}^{k|l}) \otimes \tilde{\mathcal{L}}(\Upsilon_{u|v}^{r|s})} = ((\mathcal{M}_{m|n}^{k|l} \otimes_{\mathfrak{R}} \overline{\mathcal{M}}_{m|n}^{r|s})^{\mathcal{L}(\Upsilon_{u|v}^{k|l}) \otimes \tilde{\mathcal{L}}(\Upsilon_{u|v}^{r|s})})^{U_q(\mathfrak{gl}_{m|n})}.$$

Lemma 4.14. *There are following assertions:*

(1) *As a $\mathcal{L}(U_q(\mathfrak{gl}_{u|v})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{u|v}))$ -module, $\mathcal{X}_{u|v}$ has a multiplicity-free decomposition*

$$\mathcal{X}_{u|v} = \bigoplus_{\lambda \in \Lambda_{u|v}} L_{\lambda}^{u|v} \otimes L_{\lambda}^{u|v}.$$

(2) *$\mathcal{X}_{u|v}$ is generated by X_{ab} with $a, b \in \mathbf{I}_{u|v}$.*

(3) As a $\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{r|s}))$ -module, $\mathcal{X}_{r|s}^{k|l}$ is generated by $\mathcal{X}_{u|v}$, i.e.,

$$\mathcal{X}_{r|s}^{k|l} = \left(\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{r|s})) \right) \mathcal{X}_{u|v}.$$

Proof. By (4.11) and (4.12), we have $\mathcal{X}_{u|v} = (\mathcal{M}_{m|n}^{u|v} \otimes_{\mathfrak{R}} \overline{\mathcal{M}}_{m|n}^{u|v})^{U_q(\mathfrak{gl}_{m|n})}$. Now $\mathcal{M}_{m|n}^{u|v}$ and $\overline{\mathcal{M}}_{m|n}^{u|v}$ can be viewed respectively as subalgebras of $\mathcal{M}_{m|n}$ and $\overline{\mathcal{M}}_{m|n}$ by truncation since $u \leq m$ and $v \leq n$. This reduces to the first case discussed in Section 4.2.1, leading to part (1) and part (2) directly. For part (3), it is well known that the $\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{r|s}))$ highest weight vectors of weight λ are precisely the $\mathcal{L}(U_q(\mathfrak{gl}_{u|v})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{u|v}))$ highest weight vectors of the same weight, as there is a natural embedding $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s}) \supseteq U_q(\mathfrak{gl}_{u|v}) \otimes U_q(\mathfrak{gl}_{u|v})$. Therefore, $\mathcal{X}_{r|s}^{k|l}$ is generated by $\mathcal{X}_{u|v}$ as $\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{r|s}))$ -module. \square

Now we can finish the proof of Theorem 4.6.

Proof of Theorem 4.6. Note that $\mathcal{X}_{u|v}$ is \mathbb{Z}_+ -graded by setting $\deg X_{ab} = 1$ for all $a, b \in \mathbf{I}_{u|v}$. By Lemma 4.14, we use induction on the degree N homogeneous component of $\mathcal{X}_{u|v}$ to show that

$$\left(\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{r|s})) \right) (\mathcal{X}_{u|v})_N$$

is generated by elements X_{ab} with $a \in \mathbf{I}_{k|l}, b \in \mathbf{I}_{r|s}$. It is obvious for the case $N = 1$. In general, let $XY \in \mathcal{X}_{u|v}$ with $\deg(XY) = N > 1$. Then it is clear that for any homogeneous $x \in U_q(\mathfrak{gl}_{k|l})$ and $y \in U_q(\mathfrak{gl}_{r|s})$

$$(\mathcal{L}_x \otimes \tilde{\mathcal{L}}_y)(XY) = \sum_{(x),(y)} (-1)^{[x_{(2)}][y_{(1)}] + ([x_{(2)}] + [y_{(2)}])[X]} (\mathcal{L}_{x_{(1)}} \otimes \tilde{\mathcal{L}}_{y_{(1)}})(X) (\mathcal{L}_{x_{(2)}} \otimes \tilde{\mathcal{L}}_{y_{(2)}})(Y).$$

Since both the \mathbb{Z}_+ -gradations of X and Y are less than N , our claim follows by induction. \square

4.3. Reformulation of FFT. As in classical case, we shall reformulate the polynomial FFT for $U_q(\mathfrak{gl}_{m|n})$ in terms of superalgebra homomorphism.

In view of the linear order $>$ defined in (3.4), we define monomials $X^{\mathbf{m}} = \prod_{(a,b)}^> X_{ab}^{m_{ab}}, \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}$, where the factors are arranged decreasingly in the order $>$. Recall that $\mathcal{X}_{r|s}^{k|l}$ is an invariant superalgebra of $U_q(\mathfrak{gl}_{m|n})$ -invariants.

Lemma 4.15. *Assume that $m \geq \min\{k, r\}$ and $n \geq \min\{l, s\}$. The set of monomials $\{X^{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}\}$ constitutes a \mathcal{K} -basis for the invariant superalgebra $\mathcal{X}_{r|s}^{k|l}$.*

Proof. Note that $\Lambda_{k|l} \cap \Lambda_{r|s} \subseteq \Lambda_{m|n}$ if and only if $m \geq \min\{k, r\}$ and $n \geq \min\{l, s\}$. Under our assumption, we obtain from Lemma 4.8 the multiplicity-free decomposition

$$(4.13) \quad \mathcal{X}_{r|s}^{k|l} \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|s}$$

as $\mathcal{L}(U_q(\mathfrak{gl}_{k|l})) \otimes \tilde{\mathcal{L}}(U_q(\mathfrak{gl}_{r|s}))$ -module. By relations (4.8), the monomials $X^{\mathbf{m}}, \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}$ span the invariant algebra $\mathcal{X}_{r|s}^{k|l}$. Thus, we only need to prove the linear independence of these monomials.

Let $(\mathcal{X}_{r|s}^{k|l})_N$ be the homogeneous subspace of degree N in $\mathcal{X}_{r|s}^{k|l}$. Then we obtain from (4.13)

$$\dim_{\mathcal{K}}(\mathcal{X}_{r|s}^{k|l})_N = \sum_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}, |\lambda|=N} \dim_{\mathcal{K}}(L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|s}) = \sum_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}, |\lambda|=N} \dim_{\mathcal{K}} L_{\lambda}^{k|l} \dim_{\mathcal{K}} L_{\lambda}^{r|s}.$$

Combing this and Lemma 3.7, we have

$$\dim_{\mathcal{K}}(\mathcal{X}_{r|s}^{k|l})_N = \#\{t^{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}, |\mathbf{m}| = N\} = \#\{X^{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}, |\mathbf{m}| = N\}.$$

This finishes the proof of linear independence, since $(\mathcal{X}_{r|s}^{k|l})_N$ is spanned by the monomials $X^{\mathbf{m}}, |\mathbf{m}| = N$. \square

We introduce the following auxiliary superalgebra, which will play a crucial role in our reformulation of FFT.

Definition 4.16. Let k, l, r, s be non-negative integers. We denote by $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ the quadratic superalgebra over \mathcal{K} generated by x_{ab} with $a \in \mathbf{I}_{k|l}, b \in \mathbf{I}_{r|s}$, subject to relations:

$$(4.14) \quad \begin{aligned} (\tilde{t}_{ab})^2 &= 0, & [a] + [b] &= \bar{1}, \\ \tilde{t}_{ac}\tilde{t}_{bc} &= (-1)^{([a]+[c])([b]+[c])} q_c \tilde{t}_{bc}\tilde{t}_{ac}, & a > b, \\ \tilde{t}_{ab}\tilde{t}_{ac} &= (-1)^{([a]+[b])([a]+[c])} q_a^{-1} \tilde{t}_{ac}\tilde{t}_{ab}, & b > c, \\ \tilde{t}_{ac}\tilde{t}_{bd} &= (-1)^{([a]+[c])([b]+[d])} \tilde{t}_{bd}\tilde{t}_{ac}, & a > b, c > d, \\ \tilde{t}_{ac}\tilde{t}_{bd} &= (-1)^{([a]+[c])([b]+[d])} \tilde{t}_{bd}\tilde{t}_{ac} \\ &\quad + (-1)^{[a]([b]+[d])+[b][d]} (q - q^{-1}) \tilde{t}_{bc}\tilde{t}_{ad}, & a > b, c < d. \end{aligned}$$

The \mathbb{Z}_2 -grading is given by $[\tilde{t}_{ab}] = [a] + [b]$. We shall write $\widetilde{\mathcal{M}}_{k|l} := \widetilde{\mathcal{M}}_{k|l}^{k|l}$ for convenience.

Lemma 4.17. Let k, l, r, s be fixed non-negative integers. For any $m, n \in \mathbb{Z}_+$, the \mathcal{K} -linear map from $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ to the invariant superalgebra $\mathcal{X}_{r|s}^{k|l}$ of $U_q(\mathfrak{gl}_{m|n})$ -invariants

$$(4.15) \quad \Psi_{r|s}^{k|l} : \widetilde{\mathcal{M}}_{r|s}^{k|l} \longrightarrow \mathcal{X}_{r|s}^{k|l}, \quad \tilde{t}_{ab} \longmapsto X_{ab}, \quad \forall a \in \mathbf{I}_{k|l}, b \in \mathbf{I}_{r|s}.$$

is a surjective superalgebra homomorphism.

Proof. It is clear from (4.8) and (4.14) that the map is a well defined superalgebra homomorphism. The surjectivity is immediate by Theorem 4.6. \square

Lemma 4.18. Suppose that $m \geq \min\{k, r\}$ and $n \geq \min\{l, s\}$. Then $\Psi_{r|s}^{k|l}$ is a superalgebra isomorphism.

Proof. By Lemma 4.17, it suffices to prove $\text{Ker } \Psi_{r|s}^{k|l} = 0$, which is equivalent to show that elements in $\mathcal{X}_{r|s}^{k|l}$ have no nontrivial relations except (4.8) if $m \geq \min\{k, r\}$ and $n \geq \min\{l, s\}$.

Assume that $\mathcal{X}_{r|s}^{k|l}$ has a nontrivial relation except (4.8). We denote it by

$$R(X_{a_1 b_1}, \dots, X_{a_p b_p}) = 0, \quad p \geq 1,$$

where $R(X_{a_1 b_1}, \dots, X_{a_p b_p})$ is a non-zero polynomial in $\mathcal{X}_{r|s}^{k|l}$. Using Lemma 4.15, we obtain the expression $R(X_{a_1 b_1}, \dots, X_{a_p b_p}) = \sum_{\mathbf{m}} c_{\mathbf{m}} X^{\mathbf{m}} = 0$ for finitely many $c_{\mathbf{m}} \in \mathcal{K}$ with $\mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}$. This forces all $c_{\mathbf{m}} = 0$ since $X^{\mathbf{m}}$'s are \mathcal{K} -linearly independent, and hence $R(X_{a_1 b_1}, \dots, X_{a_p b_p})$ is a zero polynomial, leading to a contradiction. This completes our proof. \square

Lemma 4.18 can be viewed as another formulation of $\widetilde{\mathcal{M}}_{r|s}^{k|l}$, that is, $\widetilde{\mathcal{M}}_{r|s}^{k|l} \cong \mathcal{X}_{r|s}^{k|l}$ as superalgebras for sufficient large m and n . Therefore, $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ acquires the $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module structure from its counterpart on $\mathcal{X}_{r|s}^{k|l}$ through the isomorphism $\Psi_{r|s}^{k|l}$.

Proposition 4.19. There are following properties of $\widetilde{\mathcal{M}}_{r|s}^{k|l}$:

(1) (Howe duality) The superalgebra $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ admits the multiplicity-free decomposition

$$\widetilde{\mathcal{M}}_{r|s}^{k|l} \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|s}$$

as $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module;

(2) (PBW basis) The set $\mathcal{T} := \{\tilde{t}^{\mathbf{m}} = \prod_{(a,b)}^{\geq} \tilde{t}_{ab}^{m_{ab}} \mid \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}\}$ constitutes a \mathcal{K} -basis for $\widetilde{\mathcal{M}}_{r|s}^{k|l}$.

Proof. By Lemma 4.18, $\widetilde{\mathcal{M}}_{r|s}^{k|l} \cong \mathcal{X}_{r|s}^{k|l}$ as superalgebras for $m \geq \min\{k, r\}$ and $n \geq \min\{l, s\}$. Then part (1) follows from (4.13) and part (2) from Lemma 4.15 and Lemma 4.18. \square

Remark 4.20. For any $\mathbf{m} = (m_{ab}), \mathbf{n} = (n_{ab}) \in \mathfrak{M}_{r|s}^{k|l}$, we set $\mathbf{m} + \mathbf{n} = (m_{ab} + n_{ab})$. By relations (4.14),

$$\tilde{t}^{\mathbf{m}} \tilde{t}^{\mathbf{n}} = (-1)^{\delta_1} q^{\delta_2} \tilde{t}^{\mathbf{m}+\mathbf{n}} + (q - q^{-1})X,$$

where $X \in \mathbb{C}[q, q^{-1}]\mathcal{T}$, and $\delta_1, \delta_2 \in \mathbb{Z}$ (which depend on \mathbf{m} and \mathbf{n}). Here $\tilde{t}^{\mathbf{m}+\mathbf{n}} = 0$ if $m_{ab} + n_{ab} = 2$ for some $[a] + [b] = \bar{1}$.

We reformulate the FFT of invariant theory for $U_q(\mathfrak{gl}_{m|n})$ as follows.

Theorem 4.21. (FFT for $U_q(\mathfrak{gl}_{m|n})$) *The superalgebra homomorphism $\Psi_{r|s}^{k|l} : \widetilde{\mathcal{M}}_{r|s}^{k|l} \rightarrow \mathcal{X}_{r|s}^{k|l}$ is surjective. Moreover, $\Psi_{r|s}^{k|l}$ is a $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{s|r})$ -module homomorphism.*

For notational convenience, we introduce the shorthand $\lambda = (1^{m_1} 2^{m_2} \dots)$ for the partition λ with m_1 's 1, m_2 's 2 and so on. For any two partitions λ and μ , $\lambda \subseteq \mu$ means that the Young diagram of λ can be embedded into that of μ .

Corollary 4.22. *As a $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module, $\text{Ker } \Psi_{r|s}^{k|l}$ admits the multiplicity-free decomposition*

$$\text{Ker } \Psi_{r|s}^{k|l} \cong \bigoplus_{\lambda_c \subseteq \lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|s}, \quad \text{with } \lambda_c = ((m+1)^{n+1}).$$

Proof. This is a consequence of Theorem 4.21, Proposition 4.19 and Lemma 4.8. \square

We close this section by giving the third formulation of $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ via braiding operator on $V^{k|l} \otimes (V^{r|s})^*$.

Proposition 4.23. *As a superalgebra, $\widetilde{\mathcal{M}}_{r|s}^{k|l} \cong S_q(V^{k|l} \otimes (V^{r|s})^*)$. Thus we have the following multiplicity-free decomposition*

$$S_q(V^{k|l} \otimes (V^{r|s})^*) \cong \bigoplus_{\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|s}$$

as $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module. In particular, $V^{k|l} \otimes (V^{r|s})^*$ is a flat $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module (in the sense of Section 3.2).

Proof. The proof is similar to that of Proposition 3.12. Let $\tilde{U} = V^{k|l} \otimes (V^{r|s})^*$ and P_{23} be graded the permutation of two middle factors in $\tilde{U} \otimes \tilde{U}$. Then we have

$$\Lambda_q^2(\tilde{U}) = P_{23} \left((S_q^2(V^{k|l}) \otimes \Lambda_q^2((V^{r|s})^*)) \oplus (\Lambda_q^2(V^{k|l}) \otimes S_q^2((V^{r|s})^*)) \right).$$

Using bases given in (3.7), (3.8) and Proposition 3.14, we obtain the quadratic relations for $S_q(\tilde{U}) = T(\tilde{U})/\Lambda_q^2(\tilde{U})$ as follows:

$$\begin{aligned} (x_{ia})^2 &= 0, & [i] + [a] &= \bar{1}, \\ x_{ja}x_{ia} &= (-1)^{([i]+[a])([j]+[a])} q_a x_{ia}x_{ja}, & j &> i, \\ x_{ib}x_{ia} &= (-1)^{([i]+[a])([i]+[b])} q_i^{-1} x_{ia}x_{ib}, & b &> a, \\ x_{jb}x_{ia} &= (-1)^{([i]+[a])([j]+[b])} x_{ia}x_{jb}, & j &> i, b > a, \\ x_{ja}x_{ib} &= (-1)^{([i]+[b])([j]+[a])} x_{ib}x_{ja} \\ &\quad + (-1)^{[i]([j]+[a])+[j][a]} (q - q^{-1}) x_{jb}x_{ia}, & j &> i, a < b. \end{aligned}$$

Here $x_{ia} = v_i \otimes v_a^*$. The isomorphism between $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ and $S_q(\tilde{U})$ is specified by $\tilde{t}_{ia} \mapsto (-1)^{[i][a]} x_{ia}$. The multiplicity-free decomposition is also clear from this isomorphism and Proposition 4.19. \square

Now the superalgebra homomorphisms given in Proposition 3.12, Proposition 3.14, Proposition 4.23 and Theorem 4.21 fit into the following commutative diagram, and hence we obtain the surjective superalgebra homomorphism in the bottom row, which reduces in the limit $q \rightarrow 1$ to the classical case.

$$\begin{array}{ccc}
\widetilde{\mathcal{M}}_{r|s}^{k|l} & \xrightarrow{\Psi_{r|s}^{k|l}} & \mathcal{X}_{r|s}^{k|l} := (\mathcal{M}_{m|n}^{k|l} \otimes_{\mathfrak{R}} \overline{\mathcal{M}}_{m|n}^{r|s})^{\mathrm{U}_q(\mathfrak{gl}_{m|n})} \\
\downarrow \cong & & \downarrow \cong \\
S_q(V^{k|l} \otimes (V^{r|s})^*) & \longrightarrow & \left(S_q(V^{k|l} \otimes V^{m|n}) \otimes_{\mathfrak{R}} S_q((V^{r|s})^* \otimes (V^{m|n})^*) \right)^{\mathrm{U}_q(\mathfrak{gl}_{m|n})}
\end{array}$$

5. THE SFT OF INVARIANT THEORY FOR $\mathrm{U}_q(\mathfrak{gl}_{m|n})$

In this section, we shall describe the kernel of the superalgebra epimorphism $\Psi_{r|s}^{k|l} : \widetilde{\mathcal{M}}_{r|s}^{k|l} \rightarrow \mathcal{X}_{r|s}^{k|l}$ as a two-sided ideal of $\widetilde{\mathcal{M}}_{r|s}^{k|l}$. The images of nonzero elements in kernel will give rise to new relations among invariants in $\mathcal{X}_{r|s}^{k|l}$, apart from the quadratic relations (4.8).

The main idea is to identify $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ as a subalgebra of $\widetilde{\mathcal{M}}_{K|L}$ with $K = \max\{k, r\}$ and $L = \max\{l, s\}$. This way we obtain $\mathrm{Ker} \Psi_{r|s}^{k|l}$ by restricting $\mathrm{Ker} \Psi_{K|L}$ to $\widetilde{\mathcal{M}}_{r|s}^{k|l}$; see commutative diagram (5.7) and Theorem 5.15. We start by developing some general results on the algebraic structure of $\mathcal{M}_{K|L}$ (resp. $\mathcal{M}_{r|s}^{k|l}$), taking advantage of some nice properties of matrix elements. Then we translate these results to $\widetilde{\mathcal{M}}_{K|L}$ (resp. $\widetilde{\mathcal{M}}_{r|s}^{k|l}$).

5.1. Algebraic structure of $\mathcal{M}_{K|L}$. We shall recall a well-known fact. Suppose that $K, L \in \mathbb{Z}_+$ and V is a left $\mathrm{U}_q(\mathfrak{gl}_{K|L})$ -module, then it has a canonical structure of a right $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule with structure map

$$\delta : V \rightarrow V \otimes \mathcal{K}[\mathrm{GL}_q(K|L)].$$

(Recall that $\mathcal{K}[\mathrm{GL}_q(K|L)] = \mathcal{M}_{K|L} \overline{\mathcal{M}}_{K|L}$, which is isomorphic to the coordinate superalgebra of $\mathrm{U}_q(\mathfrak{gl}_{K|L})$ as mentioned in Remark 2.10.) Using Sweedler's notation, we write $\delta(v) = \sum_{(v)} v_{(0)} \otimes v_{(1)}$. Then we obtain

$$\sum_{(v)} v_{(0)} \langle v_{(1)}, x \rangle = (-1)^{[x][v]} xv, \quad \forall x \in \mathrm{U}_q(\mathfrak{gl}_{K|L}), v \in V.$$

Therefore, the comodule structure induces the original module structure in the canonical way and vice versa.

Let $\lambda \in \Lambda_{K|L}$ be a (K, L) -hook partition and (L_λ, π^λ) be an irreducible representation of $\mathrm{U}_q(\mathfrak{gl}_{K|L})$ with the highest weight λ^\natural . Define the elements $t_{ab}^\lambda \in \mathrm{U}_q(\mathfrak{gl}_{K|L})^\circ$ by

$$\langle t_{ab}^\lambda, x \rangle = \pi^\lambda(x)_{ab}, \quad \forall x \in \mathrm{U}_q(\mathfrak{gl}_{K|L}), a, b = 1, 2, \dots, \dim_{\mathcal{K}} L_\lambda.$$

We write T_λ for the subspace spanned by these elements t_{ab}^λ . Then it follows from Proposition 2.2 that $T_\lambda \subset \mathcal{M}_{K|L}$, and L_λ naturally affords the right and left $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule structure, which are given respectively by

$$\begin{aligned}
\delta_\lambda^R : L_\lambda &\rightarrow L_\lambda \otimes T_\lambda, & v_b^\lambda &\mapsto \sum_{a=1}^{\dim_{\mathcal{K}} L_\lambda} v_a^\lambda \otimes t_{ab}^\lambda, \\
\delta_\lambda^L : L_\lambda &\rightarrow T_\lambda \otimes L_\lambda, & v_a^\lambda &\mapsto \sum_{b=1}^{\dim_{\mathcal{K}} L_\lambda} t_{ab}^\lambda \otimes v_b^\lambda,
\end{aligned}$$

where v_a^λ ($a = 1, 2, \dots, \dim_{\mathcal{K}} L_\lambda$) is a basis for L_λ . Note that T_λ is independent of the choices of basis for L_λ .

Lemma 5.1. *As a two-sided $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule, $T_\lambda \cong L_\lambda \otimes L_\lambda$ is irreducible and hence*

$$\mathcal{M}_{K|L} = \bigoplus_{\lambda \in \Lambda_{K|L}} T_\lambda.$$

Proof. It is an immediate consequence of Proposition 2.2 and Theorem 2.7. \square

For any two left irreducible $U_q(\mathfrak{gl}_{K|L})$ -modules L_λ and L_μ , we may form the tensor product $L_\lambda \otimes L_\mu$, which encodes the right $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule structure with

$$\delta_{\lambda \otimes \mu}^R : L_\lambda \otimes L_\mu \rightarrow L_\lambda \otimes L_\mu T_\lambda T_\mu, \quad v_b^\lambda \otimes v_d^\mu \mapsto \sum_{a,c} v_a^\lambda \otimes v_c^\mu (-1)^{[v_c^\mu][t_{ab}^\lambda]} t_{ab}^\lambda t_{cd}^\mu.$$

Therefore, the product $T_\lambda T_\mu$ is nothing but the subspace of $\mathcal{M}_{K|L}$ spanned by the matrix elements of the left $U_q(\mathfrak{gl}_{K|L})$ -module $L_\lambda \otimes L_\mu$.

Proposition 5.2. *As a two-sided $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule, $T_\lambda T_\mu$ admits the following multiplicity-free decomposition*

$$(5.1) \quad T_\lambda T_\mu = \bigoplus_{\gamma \in N_{\lambda,\mu}} T_\gamma,$$

where $N_{\lambda,\mu} = \{\gamma \mid N_{\lambda,\mu}^\gamma > 0\}$ such that $N_{\lambda,\mu}^\gamma$ is the Littlewood-Richardson coefficient appearing in the decomposition $L_\lambda \otimes L_\mu \cong \bigoplus_{\gamma \in \Lambda_{K|L}} N_{\lambda,\mu}^\gamma L_\gamma$ as left $U_q(\mathfrak{gl}_{K|L})$ -module.

Proof. This is immediate from the tensor product decomposition of $L_\lambda \otimes L_\mu$, since each T_γ is the irreducible two-sided $\mathcal{M}_{K|L}$ -comodule spanned by the matrix coefficients of L_γ , and $T_{\gamma_1} \cap T_{\gamma_2} = \emptyset$ for different $\gamma_1, \gamma_2 \in N_{\lambda,\mu}$ and $T_{\gamma_1} = T_{\gamma_2}$ if $L_{\gamma_1} \cong L_{\gamma_2}$. \square

Remark 5.3. The Littlewood-Richardson coefficient $N_{\lambda,\mu}^\gamma$ can be obtained by using super duality due to [5].

Corollary 5.4. *Let λ be a (K, L) -hook partition. Then we have*

$$T_\lambda T_{(1)} = \bigoplus_{\gamma \in N_{\lambda,(1)}} T_\gamma,$$

where $N_{\lambda,(1)}$ is the set of all (K, L) -hook Young diagrams obtained by adding one box to λ .

We denote by $\langle T_\lambda \rangle_{K|L}$ the two-side ideal in $\mathcal{M}_{K|L}$ generated by T_λ . For any two-sided ideal $I \subseteq \mathcal{M}_{K|L}$, we call I the $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule ideal if it affords two-sided $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule structure (or equivalently, I is a two-sided $U_q(\mathfrak{gl}_{K|L})$ -module). Then $\langle T_\lambda \rangle_{K|L}$ is the minimal two-sided $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule ideal containing T_λ , and every two-sided $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule ideal is the sum of ideals of the form $\langle T_\lambda \rangle_{K|L}$.

The following theorem is a quantum super analogue of [6, Theorem 4.1].

Theorem 5.5. *Let λ be a (K, L) -hook partition. Then $\langle T_\lambda \rangle_{K|L}$ admits the multiplicity-free decomposition*

$$(5.2) \quad \langle T_\lambda \rangle_{K|L} = \bigoplus_{\lambda \subseteq \gamma \in \Lambda_{K|L}} T_\gamma$$

as two-sided $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule.

Proof. By Corollary 5.4, we have $\langle T_\lambda \rangle_{K|L} \subseteq \bigoplus_{\lambda \subseteq \gamma \in \Lambda_{K|L}} T_\gamma$. Now assume that $\Lambda_{K|L} \ni \gamma \supseteq \lambda$ with $|\gamma| = |\lambda| + 1$, then it follows from Corollary 5.4 again that $T_\gamma \subset T_\lambda T_{(1)}$ and hence $T_\gamma \subset \langle T_\lambda \rangle_{K|L}$. Using induction on the size of γ , we obtain $\bigoplus_{\lambda \subseteq \gamma \in \Lambda_{K|L}} T_\gamma \subseteq \langle T_\lambda \rangle_{K|L}$. \square

We write $I_\lambda = \langle T_\lambda \rangle_{K|L}$ for short. It is an immediate consequence of Theorem 5.5 that

Corollary 5.6. *We have two facts:*

- (1) *Let $\Lambda \subseteq \Lambda_{K|L}$ be a subset of (K, L) -hook partitions. The two-sided $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule $\sum_{\gamma \in \Lambda} T_\gamma$ is an ideal in $\mathcal{M}_{K|L}$ if and only if $\gamma_1 \in \Lambda$ and $\gamma_2 \supseteq \gamma_1$ imply $\gamma_2 \in \Lambda$.*
- (2) *$I_\lambda \subseteq I_\mu$ if and only if $\lambda \supseteq \mu$.*

Following [6], we will say that a set $\Lambda \subseteq \Lambda_{K|L}$ is a D -ideal (for Diagrammatic ideal) if $\gamma_1 \in \Lambda$ and $\gamma_2 \supseteq \gamma_1$ imply $\gamma_2 \in \Lambda$. Then Theorem 5.5 and Corollary 5.6 yield

Proposition 5.7. *There is a bijective correspondence between D -ideals and two-sided $\mathcal{K}[\mathrm{GL}_q(K|L)]$ -comodule ideas of $\mathcal{M}_{K|L}$, which is given by*

$$\begin{aligned} \Lambda &\rightarrow I(\Lambda) = \sum_{\gamma \in \Lambda} T_\gamma, \quad \text{for all } D\text{-ideal } \Lambda, \\ I &\rightarrow \{\gamma \mid T_\gamma \subseteq I\}, \quad \text{for all two-sided } \mathcal{K}[\mathrm{GL}_q(K|L)]\text{-comodule ideas } I. \end{aligned}$$

5.2. Algebraic structure of $\mathcal{M}_{r|s}^{k|l}$. Suppose that k, l, r, s are non-negative integers. Let $K = \max\{k, r\}$, $L = \max\{l, s\}$ and write $\mathcal{M}_{K|L} := \mathcal{M}_{K|L}^{K|L}$ for the bialgebra generated by matrix elements t_{ab} with $a \in \mathbf{I}_{K|L}$, $b \in \mathbf{I}_{K|L}$ subject to the relations (2.9). Write

$$(5.3) \quad \hat{\mathbf{I}}_{i|j} = \{a \mid K - i + 1 \leq a \leq K + j\} \subseteq \mathbf{I}_{K|L}$$

for any $i \leq K$ and $j \leq L$. We identify $\mathcal{M}_{r|s}^{k|l}$ with the subalgebra of $\mathcal{M}_{K|L}$ generated by elements t_{ab} with $a \in \hat{\mathbf{I}}_{k|l}$, $b \in \hat{\mathbf{I}}_{r|s}$, and denote this embedding by $\iota_{r|s}^{k|l}$. Conversely, there is a \mathcal{K} -algebra retraction $\pi_{r|s}^{k|l} : \mathcal{M}_{K|L} \rightarrow \mathcal{M}_{r|s}^{k|l}$ such that $\pi_{r|s}^{k|l}(t_{ab}) = t_{ab}$ for all $a \in \hat{\mathbf{I}}_{k|l}$, $b \in \hat{\mathbf{I}}_{r|s}$, and $\pi_{r|s}^{k|l}(t_{ab}) = 0$ otherwise. Clearly, $\pi_{r|s}^{k|l} \iota_{r|s}^{k|l} = \mathrm{id}_{\mathcal{M}_{r|s}^{k|l}}$.

We have $\pi_{r|s}^{k|l}(T_\lambda) \subset \mathcal{M}_{r|s}^{k|l}$ and denote the image by ${}^\pi T_\lambda$ for simplicity. Note that ${}^\pi T_\lambda \neq 0$, unless it is contained in some ideal generated by some t_{ab} with $a \in \mathbf{I}_{K|L} \setminus \hat{\mathbf{I}}_{k|l}$ or $b \in \mathbf{I}_{K|L} \setminus \hat{\mathbf{I}}_{r|s}$. This implies that $\pi_{r|s}^{k|l}(\mathcal{M}_{K|L})$ amounts to the truncation applied to $\mathcal{M}_{K|L}$ as in Theorem 3.4. Therefore,

$$(5.4) \quad {}^\pi T_\lambda \neq 0 \text{ if and only if } \lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}.$$

It follows that as $\mathcal{L}(\mathrm{U}_q(\mathfrak{gl}_{k|l})) \otimes \mathcal{R}(\mathrm{U}_q(\mathfrak{gl}_{r|s}))$ -module ${}^\pi T_\lambda \cong L_\lambda^{k|l} \otimes L_\lambda^{r|s}$ for any $\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}$, and in this case $T_\lambda = \mathcal{L}(\mathrm{U}_q(\mathfrak{gl}_{K|L})) \otimes \mathcal{R}(\mathrm{U}_q(\mathfrak{gl}_{K|L})) {}^\pi T_\lambda$.

Proposition 5.8. *Let $\lambda, \mu \in \Lambda_{k|l} \cap \Lambda_{r|s} \subseteq \Lambda_{K|L}$. We have the multiplication formula in $\mathcal{M}_{r|s}^{k|l}$*

$${}^\pi T_\lambda {}^\pi T_\mu = \bigoplus_{\gamma \in N_{\lambda, \mu} \cap \Lambda_{k|l} \cap \Lambda_{r|s}} {}^\pi T_\gamma,$$

where $N_{\lambda, \mu}$ is defined as in Proposition 5.2.

Proof. Since $\pi_{r|s}^{k|l}$ is an algebra homomorphism, we apply it to both sides of (5.1) and hence

$${}^\pi T_\lambda {}^\pi T_\mu = \bigoplus_{\gamma \in N_{\lambda, \mu}} {}^\pi T_\gamma = \bigoplus_{\gamma \in N_{\lambda, \mu} \cap \Lambda_{k|l} \cap \Lambda_{r|s}} {}^\pi T_\gamma,$$

where the last equation is the result of (5.4). \square

We denote by $\langle {}^\pi T_\lambda \rangle_{r|s}^{k|l}$ the two-sided ideal in $\mathcal{M}_{r|s}^{k|l}$ generated by ${}^\pi T_\lambda$.

Proposition 5.9. *Let $\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s} \subseteq \Lambda_{K|L}$. Then $\langle {}^\pi T_\lambda \rangle_{r|s}^{k|l}$ as $\mathcal{L}(\mathrm{U}_q(\mathfrak{gl}_{k|l})) \otimes \mathcal{R}(\mathrm{U}_q(\mathfrak{gl}_{r|s}))$ -module admits the following multiplicity-free decomposition*

$$\langle {}^\pi T_\lambda \rangle_{r|s}^{k|l} = \bigoplus_{\lambda \subseteq \gamma \in \Lambda_{k|l} \cap \Lambda_{r|s}} {}^\pi T_\gamma.$$

Proof. Applying $\pi_{r|s}^{k|l}$ to both sides of (5.2), we obtain

$$\langle {}^\pi T_\lambda \rangle_{r|s}^{k|l} = \bigoplus_{\lambda_c \subseteq \lambda \in \Lambda_{K|L}} {}^\pi T_\lambda = \bigoplus_{\lambda_c \subseteq \lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} {}^\pi T_\lambda,$$

where the last equation follows from (5.4). \square

Remark 5.10. In particular, Proposition 5.8 and Proposition 5.9 hold in the classical limit $q \rightarrow 1$. Precisely, recall that $\mathcal{M}_{K|L}|_{q=1}$ is the superalgebra over \mathbb{C} generated by t_{ab} ($a, b \in \mathbf{I}_{K|L}$) satisfying supercommutative relations $[t_{ab}, t_{cd}] = 0$. Note that t_{ab} 's are matrix elements of the natural representation $\mathbb{C}^{K|L}$ of $U(\mathfrak{gl}_{K|L})$. All arguments in Section 5.1 and Section 5.2 remain valid in this case.

5.3. The second fundamental theorem. Now we turn to the homomorphism $\Psi_{r|s}^{k|l}$ defined in (4.15). We shall characterise the kernel of $\Psi_{r|s}^{k|l}$ as a two-sided ideal of $\widetilde{\mathcal{M}}_{r|s}^{k|l}$.

5.3.1. Algebraic structure of $\widetilde{\mathcal{M}}_{r|s}^{k|l}$. For our purpose, we will embed $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ into a larger superalgebra $\widetilde{\mathcal{M}}_{K|L}$ and then explore its algebraic properties, which are similar to those of $\mathcal{M}_{r|s}^{k|l}$ in Section 5.2.

Fix non-negative integers k, l, r, s and let $K = \max\{k, r\}$, $L = \max\{l, s\}$. Let $\widetilde{\mathcal{M}}_{K|L}$ be the superalgebra generated by \tilde{t}_{ab} with $a, b \in \mathbf{I}_{K|L}$ subject to relations (4.14). We can define the following embedding and retraction, which are similar to $\iota_{r|s}^{k|l}$ and $\pi_{r|s}^{k|l}$ in Section 5.2,

$$(5.5) \quad \tilde{\iota}_{r|s}^{k|l} : \widetilde{\mathcal{M}}_{r|s}^{k|l} \rightarrow \widetilde{\mathcal{M}}_{K|L}, \quad \tilde{\pi}_{r|s}^{k|l} : \widetilde{\mathcal{M}}_{K|L} \rightarrow \widetilde{\mathcal{M}}_{r|s}^{k|l},$$

which satisfy $\tilde{\pi}_{r|s}^{k|l} \tilde{\iota}_{r|s}^{k|l} = \text{id}_{\widetilde{\mathcal{M}}_{r|s}^{k|l}}$. Notice that $\widetilde{\mathcal{M}}_{K|L}$ encodes the $U_q(\mathfrak{gl}_{K|L}) \otimes U_q(\mathfrak{gl}_{K|L})$ -module structure given in Section 4.3, which naturally induces the $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module structure on the subalgebra $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ of $\widetilde{\mathcal{M}}_{K|L}$.

Recall that $\widetilde{\mathcal{M}}_{K|L}$ admits the multiplicity-free decomposition $\widetilde{\mathcal{M}}_{K|L} \cong \bigoplus_{\lambda \in \Lambda_{K|L}} L_{\lambda}^{K|L} \otimes L_{\lambda}^{K|L}$ as $U_q(\mathfrak{gl}_{K|L}) \otimes U_q(\mathfrak{gl}_{K|L})$ -module by setting $k = r = K$ and $l = s = L$ in Proposition 4.19.

Definition 5.11. For each $\lambda \in \Lambda_{K|L}$, we denote by \tilde{T}_{λ} the subspace of $\widetilde{\mathcal{M}}_{K|L}$ which is isomorphic to $L_{\lambda}^{K|L} \otimes L_{\lambda}^{K|L}$ as $U_q(\mathfrak{gl}_{K|L}) \otimes U_q(\mathfrak{gl}_{K|L})$ -module. Thus, $\widetilde{\mathcal{M}}_{K|L} = \bigoplus_{\lambda \in \Lambda_{K|L}} \tilde{T}_{\lambda}$.

By (5.5), we have $\tilde{\pi}_{r|s}^{k|l}(\tilde{T}_{\lambda}) \subset \widetilde{\mathcal{M}}_{r|s}^{k|l}$ and denote the image by $\pi \tilde{T}_{\lambda}$ for simplicity. It is readily verified that $\tilde{\pi}_{r|s}^{k|l}(\widetilde{\mathcal{M}}_{K|L})$ amounts to the truncation applied to $\widetilde{\mathcal{M}}_{K|L}$ as in Theorem 3.4; therefore we obtain

$$(5.6) \quad \pi \tilde{T}_{\lambda} \neq 0 \text{ if and only if } \lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}.$$

In this case, $\pi \tilde{T}_{\lambda} \cong L_{\lambda}^{k|l} \otimes L_{\lambda}^{r|s} \cong \pi T_{\lambda}$ as $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module.

Observe that $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ and $\mathcal{M}_{r|s}^{k|l}$ are isomorphic as $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -modules by Theorem 3.4 and Proposition 4.19. We expect that the subspaces $\pi \tilde{T}_{\lambda}$ have the same properties as in Proposition 5.8 and Proposition 5.9, though $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ and $\mathcal{M}_{r|s}^{k|l}$ have different defining relations in general. This is given in the following lemma.

Lemma 5.12. Let $\lambda, \mu \in \Lambda_{k|l} \cap \Lambda_{r|s} \subseteq \Lambda_{K|L}$. We have the multiplication formula in $\widetilde{\mathcal{M}}_{r|s}^{k|l}$

$$\pi \tilde{T}_{\lambda} \pi \tilde{T}_{\mu} = \bigoplus_{\gamma \in N_{\lambda, \mu} \cap \Lambda_{k|l} \cap \Lambda_{r|s}} \pi \tilde{T}_{\gamma},$$

where $N_{\lambda, \mu}$ is defined as in Proposition 5.2.

Proof. It suffices to prove that $\tilde{T}_{\lambda} \tilde{T}_{\mu} = \bigoplus_{\gamma \in N_{\lambda, \mu}} \tilde{T}_{\gamma}$ in $\widetilde{\mathcal{M}}_{K|L}$, since in this case we will obtain the desired decomposition in $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ by applying $\tilde{\pi}_{r|s}^{k|l}$ to both sides of equation.

As a $U_q(\mathfrak{gl}_{K|L}) \otimes U_q(\mathfrak{gl}_{K|L})$ -module, $\tilde{T}_{\lambda} \tilde{T}_{\mu}$ is a quotient of $L_{\lambda}^{K|L} \otimes L_{\mu}^{K|L} \otimes L_{\lambda}^{K|L} \otimes L_{\mu}^{K|L}$, which admits the decomposition

$$L_{\lambda}^{K|L} \otimes L_{\mu}^{K|L} \otimes L_{\lambda}^{K|L} \otimes L_{\mu}^{K|L} \cong \bigoplus_{\nu, \gamma \in N_{\lambda, \mu}} N_{\lambda, \mu}^{\nu} N_{\lambda, \mu}^{\gamma} L_{\nu}^{K|L} \otimes L_{\gamma}^{K|L},$$

where $N_{\lambda,\mu}^\nu, N_{\lambda,\mu}^\gamma > 0$ are Littlewood-Richardson coefficients. On the other hand, $\tilde{T}_\lambda \tilde{T}_\mu$ is a submodule of $\tilde{\mathcal{M}}_{K|L} \cong \bigoplus_{\lambda \in \Lambda_{K|L}} L_\lambda^{K|L} \otimes L_\lambda^{K|L}$. It follows that $\tilde{T}_\lambda \tilde{T}_\mu$ decomposes as $\bigoplus_{\gamma \in N_{\lambda,\mu}} (L_\gamma^{K|L} \otimes L_\gamma^{K|L})^{\oplus m_\gamma}$, where m_γ ($0 \leq m_\gamma \leq 1$) denotes the multiplicity. Thus $\tilde{T}_\lambda \tilde{T}_\mu = \bigoplus_{\gamma \in N_{\lambda,\mu}} (\tilde{T}_\gamma)^{\oplus m_\gamma}$, and $\dim(\tilde{T}_\lambda \tilde{T}_\mu) = \sum_{\gamma \in N_{\lambda,\mu}} m_\gamma \dim \tilde{T}_\gamma$. It remains to show that $m_\gamma = 1$ for all $\gamma \in N_{\lambda,\mu}$.

Set $A = \mathbb{C}[q, q^{-1}]$. Let $\tilde{\mathcal{M}}_{K|L,A}$ be the superalgebra over A generated by the \tilde{t}_{ab} 's with the relations (4.14). Similarly we let $\mathcal{M}_{K|L,A}$ be the superalgebra over A generated by the t_{ab} 's. These superalgebras have PBW bases (cf. Proposition 3.8 and Proposition 4.19). Their specialisations at $q = 1$ are isomorphic to the super polynomial algebra generated by $K^2 + L^2$ even variables and $2KL$ Grassmannian variables. These are isomorphisms of $U(\mathfrak{gl}_{K|L}) \otimes U(\mathfrak{gl}_{K|L})$ -module superalgebras. Let $\tilde{T}_{\lambda,A} = \tilde{T}_\lambda \cap \tilde{\mathcal{M}}_{K|L,A}$ and $T_{\lambda,A} = T_\lambda \cap \mathcal{M}_{K|L,A}$. Denote by $\tilde{T}_{\lambda|q=1}$ the specialisation of $\tilde{T}_{\lambda,A}$ at $q = 1$, and by $T_{\lambda|q=1}$ that of $T_{\lambda,A}$. It then follows that $\tilde{T}_{\lambda|q=1} \cong T_{\lambda|q=1}$ as $U(\mathfrak{gl}_{K|L}) \otimes U(\mathfrak{gl}_{K|L})$ -modules.

For any λ and μ , it is clear from Remark 4.20 that the specialisation of $\tilde{T}_{\lambda,A} \tilde{T}_{\mu,A}$ at $q = 1$ is equal to $\tilde{T}_{\lambda|q=1} \tilde{T}_{\mu|q=1}$. By Proposition 5.2 and Remark 5.10, we have

$$\tilde{T}_{\lambda|q=1} \tilde{T}_{\mu|q=1} = \bigoplus_{\gamma \in N_{\lambda,\mu}} \tilde{T}_\gamma|_{q=1} \cong \bigoplus_{\gamma \in N_{\lambda,\mu}} L_\gamma^{K|L}|_{q=1} \otimes L_\gamma^{K|L}|_{q=1}.$$

This leads to

$$\dim_{\mathcal{K}}(\tilde{T}_\lambda \tilde{T}_\mu) \geq \sum_{\gamma \in N_{\lambda,\mu}} (\dim_{\mathbb{C}} L_\gamma^{K|L}|_{q=1})^2 = \sum_{\gamma \in N_{\lambda,\mu}} (\dim_{\mathcal{K}} L_\gamma^{K|L})^2 = \sum_{\gamma \in N_{\lambda,\mu}} \dim \tilde{T}_\gamma,$$

forcing $m_\gamma = 1$ for all $\gamma \in N_{\lambda,\mu}$. This completes the proof. \square

Let $\langle \pi \tilde{T}_\lambda \rangle_{r|s}^{k|l}$ be the two-sided ideal in $\tilde{\mathcal{M}}_{r|s}^{k|l}$ generated by $\pi \tilde{T}_\lambda$. We immediately obtain

Proposition 5.13. *Let $\lambda \in \Lambda_{k|l} \cap \Lambda_{r|s} \subseteq \Lambda_{K|L}$. Then $\langle \pi \tilde{T}_\lambda \rangle_{r|s}^{k|l}$ as $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module admits the following multiplicity-free decomposition*

$$\langle \pi \tilde{T}_\lambda \rangle_{r|s}^{k|l} = \bigoplus_{\lambda \subseteq \gamma \in \Lambda_{k|l} \cap \Lambda_{r|s}} \pi \tilde{T}_\gamma.$$

Proof. We only need to show that $\langle \tilde{T}_\lambda \rangle_{K|L} = \bigoplus_{\lambda \subseteq \gamma \in \Lambda_{K|L}} \tilde{T}_\gamma$ in $\tilde{\mathcal{M}}_{K|L}$, since our proposition follows by applying the algebra homomorphism $\tilde{\pi}_{r|s}^{k|l}$ to both sides of the equation. The decomposition of the two-sided ideal $\langle \tilde{T}_\lambda \rangle_{K|L}$ can be derived from Lemma 5.12 in a similar the way as in Theorem 5.5. \square

5.3.2. Formulation of the SFT. Let $\mathcal{X}_{K|L} := (\mathcal{M}_{m|n}^{K|L} \otimes \overline{\mathcal{M}}_{m|n}^{K|L})^{U_q(\mathfrak{gl}_{m|n})}$ be the $U_q(\mathfrak{gl}_{m|n})$ -invariant subalgebra, which is generated by X_{ab} with $a, b \in \mathbf{I}_{K|L}$ obeying relations (4.8). Identifying $\mathcal{X}_{r|s}^{k|l}$ as a subalgebra of $\mathcal{X}_{K|L}$, we immediately obtain the embedding $\tau_{r|s}^{k|l} : \mathcal{X}_{r|s}^{k|l} \rightarrow \mathcal{X}_{K|L}$ and the retraction $\psi_{r|s}^{k|l} : \mathcal{X}_{K|L} \rightarrow \mathcal{X}_{r|s}^{k|l}$ with $\psi_{r|s}^{k|l} \tau_{r|s}^{k|l} = \text{id}_{\mathcal{X}_{r|s}^{k|l}}$, which are similar to $\iota_{r|s}^{k|l}$ and $\pi_{r|s}^{k|l}$ defined in Section 5.2. The algebra homomorphisms fit into the following commutative diagram.

$$(5.7) \quad \begin{array}{ccccc} \tilde{\mathcal{M}}_{r|s}^{k|l} & \xrightarrow{\tilde{\iota}_{r|s}^{k|l}} & \tilde{\mathcal{M}}_{K|L} & \xrightarrow{\tilde{\pi}_{r|s}^{k|l}} & \tilde{\mathcal{M}}_{r|s}^{k|l} \\ \downarrow \Psi_{r|s}^{k|l} & & \downarrow \Psi_{K|L} & & \downarrow \Psi_{r|s}^{k|l} \\ \mathcal{X}_{r|s}^{k|l} & \xrightarrow{\tau_{r|s}^{k|l}} & \mathcal{X}_{K|L} & \xrightarrow{\psi_{r|s}^{k|l}} & \mathcal{X}_{r|s}^{k|l} \end{array}$$

Lemma 5.14. *Let $\lambda_c = ((m+1)^{n+1})$, then as $U_q(\mathfrak{gl}_{K|L}) \otimes U_q(\mathfrak{gl}_{K|L})$ -module,*

$$\text{Ker } \Psi_{K|L} = \bigoplus_{\lambda_c \subseteq \lambda \in \Lambda_{K|L}} \tilde{T}_\lambda = \langle \tilde{T}_{\lambda_c} \rangle_{K|L}.$$

Proof. This is a special case of Corollary 4.22 and Proposition 5.13 with $k = r = K$ and $l = s = L$. \square

Now we obtain the second fundamental theorem of invariant theory for $U_q(\mathfrak{gl}_{m|n})$ as follows.

Theorem 5.15. (SFT for $U_q(\mathfrak{gl}_{m|n})$) *Let $\Psi_{r|s}^{k|l} : \widetilde{\mathcal{M}}_{r|s}^{k|l} \rightarrow \mathcal{X}_{r|s}^{k|l}$ be the surjective superalgebra homomorphism.*

(1) *As a two-sided ideal in $\widetilde{\mathcal{M}}_{r|s}^{k|l}$,*

$$\text{Ker } \Psi_{r|s}^{k|l} = \bigoplus_{\lambda_c \subseteq \lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} \pi \tilde{T}_\lambda = \langle \pi \tilde{T}_{\lambda_c} \rangle_{r|s}^{k|l},$$

where $\langle \pi \tilde{T}_{\lambda_c} \rangle_{r|s}^{k|l}$ is the two-sided ideal in $\widetilde{\mathcal{M}}_{r|s}^{k|l}$ generated by $\pi \tilde{T}_{\lambda_c}$ with $\lambda_c = ((m+1)^{n+1})$.

(2) *$\Psi_{r|s}^{k|l}$ is an isomorphism if and only if $m \geq \min\{k, r\}$ and $n \geq \min\{l, s\}$. In this case, $\mathcal{X}_{r|s}^{k|l}$ is a quadratic superalgebra generated by X_{ab} , $a \in \mathbf{I}_{k|l}$, $b \in \mathbf{I}_{r|s}$ subject to relations (4.8), and it has a monomial basis $\{X^{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}\}$.*

Proof. Let $K = \max\{k, r\}$ and $L = \max\{l, s\}$, we have

$$\begin{aligned} \text{Ker } \Psi_{r|s}^{k|l} &= \pi_{r|s}^{k|l}(\text{Ker } \Psi_{K|L}) && \text{by commutative diagram (5.7),} \\ &= \bigoplus_{\lambda_c \subseteq \lambda \in \Lambda_{K|L}} \pi \tilde{T}_\lambda && \text{by Lemma 5.14,} \\ &= \bigoplus_{\lambda_c \subseteq \lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} \pi \tilde{T}_\lambda && \text{by (5.6),} \\ &= \langle \pi \tilde{T}_{\lambda_c} \rangle_{r|s}^{k|l} && \text{by Proposition 5.13.} \end{aligned}$$

For part (2), it is clear from part (1) that $\text{Ker } \Psi_{r|s}^{k|l} = 0$ if and only if $\Lambda_{k|l} \cap \Lambda_{r|s} \subseteq \Lambda_{m|n}$ if and only if $m \geq \min\{k, r\}$ and $n \geq \min\{l, s\}$, whence the isomorphism follows. In this case, the monomial basis is given by Lemma 4.15. \square

Remark 5.16. It will be very interesting to find the elements which generate $\pi \tilde{T}_{\lambda_c}$, thus to make the SFT more explicit. We will do this for the cases $U_q(\mathfrak{gl}_m)$ and $U(\mathfrak{gl}_{m|n})$ in Section 6.

6. APPLICATIONS

In this section, we shall elucidate how our main results can be applied to derive polynomial versions of invariant theory for $U_q(\mathfrak{gl}_m)$ and $U(\mathfrak{gl}_{m|n})$. This contextualises and simplifies the classical treatments in [16, 27, 28].

6.1. Invariant theory for $U_q(\mathfrak{gl}_m)$. This is a special case of $U_q(\mathfrak{gl}_{m|n})$ with $n = 0$, for which the FFT of invariant theory was obtained in [16]. We shall work with the following quantum analogue of polynomial algebra

$$\mathcal{P}_{k,r} := \mathcal{M}_{k,m} \otimes_{\mathfrak{R}} \overline{\mathcal{M}}_{r,m},$$

where $\mathcal{M}_{k,m} := \mathcal{M}_{m|0}^{k|0}$, $\overline{\mathcal{M}}_{r,m} := \overline{\mathcal{M}}_{m|0}^{r|0}$. By Remark 4.2 $\mathcal{P}_{k,r}$ is the flat deformation of the symmetric algebra $S(\mathbb{C}^k \otimes \mathbb{C}^m \oplus \mathbb{C}^r \otimes (\mathbb{C}^m)^*)$, and is generated by elements T_{ai} and \bar{T}_{bj} ($1 \leq a \leq k, 1 \leq b \leq r, 1 \leq i, j \leq m$) satisfying relations (4.1) with all the parities of indexes being even.

Let $\mathcal{X}_{k,r} := (\mathcal{P}_{k,r})^{U_q(\mathfrak{gl}_m)}$ be the $U_q(\mathfrak{gl}_m)$ -invariant subalgebra of $\mathcal{P}_{k,r}$. Then by Lemma 4.4 the elements $X_{ab} = \sum_{i=1}^m T_{ai} \bar{T}_{bi}$ belong to $\mathcal{X}_{k,r}$, $\forall 1 \leq a \leq k, 1 \leq b \leq r$. It follows from Lemma 4.5 that

Lemma 6.1. ([16, Lemma 6.9]) *The elements $X_{ab} \in \mathcal{X}_{k,r}$ are subject to the following relations*

$$\begin{aligned} X_{ac} X_{bc} &= q X_{bc} X_{ac}, & a > b, \\ X_{ab} X_{ac} &= q^{-1} X_{ac} X_{ab}, & b > c, \\ X_{ac} X_{bd} &= X_{bd} X_{ac}, & a > b, c > d, \\ X_{ac} X_{bd} &= X_{bd} X_{ac} + (q - q^{-1}) X_{bc} X_{ad}, & a > b, c < d. \end{aligned}$$

Let $\widetilde{\mathcal{M}}_{k,r} := \widetilde{\mathcal{M}}_{r|0}^{k|0}$ be the algebra as defined in Definition 4.16. It is easy to verify that $\mathcal{M}_{k,r} \cong \widetilde{\mathcal{M}}_{k,r}$ as algebras, which is given by $t_{ab} \mapsto \tilde{t}_{a,r+1-b}$. The algebra homomorphism defined in (4.15) now can be reformulated as

$$\Psi_{k,r} : \mathcal{M}_{k,r} \rightarrow \mathcal{X}_{k,r}, \quad t_{ab} \mapsto X_{a,r+1-b}.$$

By Theorem 4.21, we obtain the FFT of invariant theory for $U_q(\mathfrak{gl}_m)$ as follows.

Theorem 6.2. (FFT for $U_q(\mathfrak{gl}_m)$) *The algebra homomorphism $\Psi_{k,r} : \mathcal{M}_{k,r} \rightarrow \mathcal{X}_{k,r}$ is surjective.*

Now suppose that $m < \min\{k, r\}$. We claim that $\Psi_{k,r}$ is a surjective $U_q(\mathfrak{gl}_k)$ -module homomorphism, since it is straightforward to verify that

$$\Psi_{k,r}(\mathcal{L}_x(f)) = \mathcal{L}_x(\Psi_{k,r}(f)), \quad \forall x \in U(\mathfrak{gl}_{k|l}), f \in \mathcal{M}_{k,r},$$

where the action \mathcal{L}_x on $\mathcal{X}_{k,r}$ is inherited from that on $\mathcal{P}_{k,r}$. By Theorem 3.4 and Lemma 4.8, $\text{Ker } \Psi_{k,r}$ as $U(\mathfrak{gl}_k) \otimes U(\mathfrak{gl}_r)$ -module admits the following multiplicity-free decomposition

$$\text{Ker } \Psi_{k,r} \cong \bigoplus_{(m+1) \subseteq \lambda \in \Lambda_{k|0} \cap \Lambda_{r|0}} L_{\lambda}^k \otimes L_{\lambda}^r.$$

Let $K = \max\{k, r\}$ and $T_{(m+1)}$ be the subspace of $\mathcal{M}_{K,K}$ spanned by the matrix elements of the irreducible $U_q(\mathfrak{gl}_K)$ -module $L_{(m+1)}^K$. It follows from Theorem 5.15 that as a two-sided ideal of $\mathcal{M}_{k,r}$,

$$(6.1) \quad \text{Ker } \Psi_{k,r} = \langle {}^{\pi}T_{(m+1)} \rangle_{k,r},$$

where $\pi_{k,r}$ is the algebra retraction $\pi_{k,r} : \mathcal{M}_{K,K} \rightarrow \mathcal{M}_{k,r}$ such that $\pi_{k,r}(t_{ab}) = t_{ab}$ if $1 \leq a \leq k$ and $1 \leq b \leq r$, and 0 otherwise.

We now take a closer look at $L_{(m+1)}^K$ and find corresponding matrix elements. Recall that quantum exterior algebra $\Lambda_q(V^K)$ is generated by the elements $\xi_a, 1 \leq a \leq K$ subject to the relations

$$\xi_a^2 = 0, \quad \forall a, \quad \xi_a \xi_b = -q^{-1} \xi_b \xi_a, \quad a > b.$$

The quantum skew Howe duality in Remark 3.13 gives rise to the multiplicity-free decomposition (see also [16, Theorem 6.16])

$$\Lambda_q(V^K) = \bigoplus_{N=0}^K \Lambda_q(V^K)_N,$$

where $\Lambda_q(V^K)_N \cong L_{(N)}^K$ is the irreducible $U_q(\mathfrak{gl}_K)$ -module with the highest weight $\sum_{i=1}^N \epsilon_i$, for which the basis is given by

$$(6.2) \quad \xi_{\underline{a}} := \xi_{a_1} \xi_{a_2} \cdots \xi_{a_N}, \quad \underline{a} = (a_1, a_2, \dots, a_N) \text{ with } 1 \leq a_1 < a_2 < \cdots < a_N \leq K.$$

Observe that $\Lambda_q(V^K)$ is a $\text{GL}_q(K) = \mathcal{M}_{K,K} \overline{\mathcal{M}}_{K,K}$ -comodule with the structure map

$$\delta : \Lambda_q(V^K) \rightarrow \text{GL}_q(K) \otimes \Lambda_q(V^K), \quad \xi_a \mapsto \sum_{b=1}^K t_{ab} \otimes \xi_b.$$

In particular, for any two sequences $\underline{a}, \underline{b}$ as (6.2), $\delta(\xi_{\underline{a}}) = \sum_{\underline{b}} \Delta(\underline{a}, \underline{b}) \otimes \xi_{\underline{b}}$, yielding the matrix elements $\Delta(\underline{a}, \underline{b})$ of $\Lambda_q(V^K)_N \cong L_{(N)}^K$. We call $\Delta(\underline{a}, \underline{b})$ *quantum minor*, which can be written explicitly as

$$(6.3) \quad \Delta(\underline{a}, \underline{b}) = \sum_{\sigma \in \text{Sym}_N} (-q^{-1})^{\ell(\sigma)} t_{a_1, b_{\sigma(1)}} \cdots t_{a_N, b_{\sigma(N)}},$$

where ℓ is the length function of the symmetric group Sym_N .

We now arrive at the the SFT of invariant theory for $U_q(\mathfrak{gl}_m)$ by Theorem 5.15.

Theorem 6.3. (SFT for $U_q(\mathfrak{gl}_m)$) *Let $\Psi_{k,r} : \mathcal{M}_{k,r} \rightarrow \mathcal{X}_{k,r}$ be the surjective algebra homomorphism.*

(1) *As a two-sided ideal of $\mathcal{M}_{k,r}$, $\text{Ker } \Psi_{k,r}$ is generated by quantum minors $\Delta(\underline{a}, \underline{b})$ with*

$$\begin{aligned} \underline{a} &= (a_1, a_2, \dots, a_{m+1}), & 1 \leq a_1 < a_2 < \cdots < a_{m+1} \leq k, \\ \underline{b} &= (b_1, b_2, \dots, b_{m+1}), & 1 \leq b_1 < b_2 < \cdots < b_{m+1} \leq r. \end{aligned}$$

- (2) $\Psi_{k,r}$ is an isomorphism if and only if $m \geq \min\{k, r\}$. In this case, $\mathcal{X}_{k,r}$ is a quadratic algebra generated by $X_{ab}, 1 \leq a \leq k, 1 \leq b \leq r$ subject to relations in Lemma 6.1, and it has a monomial basis $\{\prod_{(a,b)}^> X_{a+r+1-b}^{m_{ab}} \mid \mathbf{m} \in \mathfrak{M}_{r|0}^{k|0}\}$.

Proof. We only need to prove part (1). By (6.1), it suffices to consider the generators of ${}^\pi T_{(m+1)}$. Note that $T_{(m+1)}$ in $\mathcal{M}_{K,K}$ is spanned by quantum minors of the form $\Delta(\underline{a}, \underline{b})$ in (6.3) with $\underline{a}, \underline{b} \in \mathbb{Z}_+^{m+1}$ as in (6.2). Thus, ${}^\pi T_{(m+1)}$ is spanned by these $\Delta(\underline{a}, \underline{b})$ with $\underline{a}, \underline{b}$ as required in the theorem, otherwise they will be killed by $\pi_{k,r}$. The monomial basis for $\mathcal{X}_{k,r}$ is given by $\Psi_{k,r}(t^{\mathbf{m}}) = \prod_{(a,b)}^> X_{a+r+1-b}^{m_{ab}}, \mathbf{m} \in \mathfrak{M}_{r|0}^{k|0}$ when $m \geq \min\{k, r\}$. \square

6.2. Invariant theory for $U(\mathfrak{gl}_{m|n})$. Recall that $U(\mathfrak{gl}_{m|n})$ is the universal enveloping algebra of the general lineal Lie superalgebra $\mathfrak{gl}_{m|n}$. It has the structure of supercocomutative Hopf superalgebra, with comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$, counit $\epsilon(x) = 0$ and antipode $S(x) = -x, x \in \mathfrak{gl}_{m|n}$.

We shall work with the following supercommutative polynomial algebra

$$\mathcal{P}_{r|s}^{k|l}|_{q=1} := \mathcal{M}_{m|n}^{k|l}|_{q=1} \otimes_{\mathbb{C}} \overline{\mathcal{M}}_{m|n}^{r|s}|_{q=1} \cong S(\mathbb{C}^{k|l} \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{r|s} \otimes (\mathbb{C}^{m|n})^*),$$

which is generated by T_{ai} and \bar{T}_{bj} subject to relations

$$\begin{aligned} T_{ai}^2 &= \bar{T}_{bj}^2 = 0, & [a] + [i] &= [b] + [j] = \bar{1}, \\ T_{ai}T_{ck} &= (-1)^{([a]+[i])([c]+[k])} T_{ck}T_{ai}, \\ \bar{T}_{bj}\bar{T}_{dl} &= (-1)^{([b]+[j])([d]+[l])} \bar{T}_{dl}\bar{T}_{bj}, \\ T_{ai}\bar{T}_{bj} &= (-1)^{([a]+[i])([b]+[j])} \bar{T}_{bj}T_{ai}, \end{aligned}$$

for all $a, c \in \mathbf{I}_{k|l}, b, d \in \mathbf{I}_{r|s}$ and $i, j, k, l \in \mathbf{I}_{m|n}$. We note that the arguments in Section 2.4 remain valid in classical case, replacing q by 1 (see, e.g. [29]). Thus, $U(\mathfrak{gl}_{k|l}), U(\mathfrak{gl}_{r|s})$ and $U(\mathfrak{gl}_{m|n})$ act on these generators T_{ai} and \bar{T}_{bj} as the way in the quantum case, and hence the elements

$$X_{ab} = \sum_{i \in \mathbf{I}_{m|n}} (-1)^{[a]([b]+[i])} T_{ai}\bar{T}_{bi}, \quad \forall a \in \mathbf{I}_{k|l}, b \in \mathbf{I}_{r|s}$$

belong to $U(\mathfrak{gl}_{m|n})$ -invariant subalgebra $\mathcal{X}_{r|s}^{k|l}|_{q=1}$ of $\mathcal{P}_{r|s}^{k|l}|_{q=1}$ by Lemma 4.4. The following can be verified directly as that in Lemma 4.5.

Lemma 6.4. *The elements $X_{ab} \in \mathcal{X}_{r|s}^{k|l}|_{q=1}$ satisfy*

$$X_{ab}^2 = 0, [a] + [b] = \bar{1}, \quad X_{ab}X_{cd} = (-1)^{([a]+[b])([c]+[d])} X_{cd}X_{ab}.$$

for any $a, c \in \mathbf{I}_{k|l}, b, d \in \mathbf{I}_{r|s}$

Since $\widetilde{\mathcal{M}}_{r|s}^{k|l}|_{q=1} \cong \mathcal{M}_{r|s}^{k|l}|_{q=1}$ as superalgebras, we obtain from Theorem 4.21 the FFT of invariant theory for $U(\mathfrak{gl}_{m|n})$ as follows.

Theorem 6.5. (FFT for $U(\mathfrak{gl}_{m|n})$) *The superalgebra homomorphism*

$$\Psi_{r|s}^{k|l} : \mathcal{M}_{r|s}^{k|l}|_{q=1} \rightarrow \mathcal{X}_{r|s}^{k|l}|_{q=1}, \quad t_{ab} \mapsto X_{ab}$$

is surjective. Moreover, $\Psi_{r|s}^{k|l}$ is a $U_q(\mathfrak{gl}_{k|l}) \otimes U_q(\mathfrak{gl}_{r|s})$ -module homomorphism.

Now by Theorem 3.4 and Lemma 4.8, we obtain

$$\text{Ker } \Psi_{r|s}^{k|l} \cong \bigoplus_{\lambda_c \subseteq \lambda \in \Lambda_{k|l} \cap \Lambda_{r|s}} L_{\lambda}^{k|l}|_{q=1} \otimes L_{\lambda}^{r|s}|_{q=1}, \quad \text{with } \lambda_c = ((m+1)^{n+1})$$

as $U(\mathfrak{gl}_{k|l}) \otimes U(\mathfrak{gl}_{r|s})$ -module. Let $K = \max\{k, r\}, L = \max\{l, s\}$ and $T_{\lambda_c}|_{q=1}$ be the subspace of $\mathcal{M}_{K|L}|_{q=1}$ spanned by the matrix elements of the irreducible $U(\mathfrak{gl}_{K|L})$ -module $L_{\lambda_c}^{K|L}|_{q=1}$. By Theorem 5.15, we obtain that as a two-sided ideal of $\mathcal{M}_{r|s}^{k|l}|_{q=1}$,

$$(6.4) \quad \text{Ker } \Psi_{r|s}^{k|l} = \langle {}^\pi T_{\lambda_c}|_{q=1} \rangle_{r|s}^{k|l},$$

where $\pi_{r|s}^{k|l}$ is the algebra retraction $\pi_{r|s}^{k|l} : \mathcal{M}_{K|L}|_{q=1} \rightarrow \mathcal{M}_{r|s}^{k|l}|_{q=1}$ such that $\pi_{r|s}^{k|l}(t_{ab}) = t_{ab}$ if $a \in \hat{\mathbf{I}}_{k|l}$ and $b \in \hat{\mathbf{I}}_{r|s}$, and 0 otherwise. Here $\hat{\mathbf{I}}_{k|l}, \hat{\mathbf{I}}_{r|s}$ are defined as in (5.3).

It remains to figure out the matrix elements of $L_{\lambda_c}^{K|L}|_{q=1}$. We start by briefly recalling the Schur-Weyl duality of $\mathfrak{gl}_{K|L}$ due to [1] (also see [28]). Let $\{v_i \mid i \in \mathbf{I}_{K|L}\}$ be a basis for $\mathbb{C}^{K|L}$ and write $v_I := v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_N} \in (\mathbb{C}^{K|L})^{\otimes N}$ with $I = (i_1, i_2, \dots, i_k)$. The elementary transposition $(a, a+1) \in \text{Sym}_N$ acts on v_I by permuting v_{i_a} and $v_{i_{a+1}}$ together with a sign $(-1)^{[v_{i_a}][v_{i_{a+1}}]}$. This in general induces the action

$$\sigma.v_I = c(I, \sigma^{-1})v_{\sigma I}, \quad \text{with } c(I, \sigma) = \pm 1, \quad \forall \sigma \in \text{Sym}_N,$$

where σ acts on the sequence I naturally. The Schur-Weyl duality gives the multiplicity-free decomposition of the tensor product

$$(\mathbb{C}^{K|L})^{\otimes N} \cong \bigoplus_{\lambda \in \Lambda_{K|L}, |\lambda|=N} S^\lambda \otimes L_\lambda^{K|L}|_{q=1}$$

as $\mathbb{C}\text{Sym}_N \otimes U(\mathfrak{gl}_{K|L})$ -module, where S^λ stands for the irreducible $\mathbb{C}\text{Sym}_N$ -module associated to λ .

We recall some combinatorics on tableaux before giving a nice basis for $L_\lambda^{K|L}|_{q=1}$. A *standard λ -tableau* is obtained by filling the Young diagram of λ with integers $1, \dots, N$ such that these entries increase down columns and along rows. We denote by $R(\mathbf{t})$ and $C(\mathbf{t})$ respectively the row and column stabilisers in Sym_N . Define the Young symmetriser on \mathbf{t} by

$$(6.5) \quad y_{\mathbf{t}} := \sum_{\sigma \in R(\mathbf{t}), \tau \in C(\mathbf{t})} (-1)^{\ell(\tau)} \sigma \tau.$$

Let I be a sequence of elements from $\mathbf{I}_{K|L}$ of length N . A \mathbf{t} -semistandard sequence I is obtained by filling the tableau \mathbf{t} with elements from I , replacing each element a by $i_a \in I$ ($1 \leq a \leq N$), in such a way that the elements of \mathbf{t} do not decrease from left to right and downward, the even elements strictly increase along columns, while the odd elements strictly increase along rows. The following result is well known [1, 28].

Proposition 6.6. *Let $\lambda \in \Lambda_{K|L}$ with size $|\lambda| = N$ and \mathbf{t} be any fixed standard λ -tableau, $y_{\mathbf{t}}(\mathbb{C}^{K|L})^{\otimes N} \cong L_\lambda^{K|L}|_{q=1}$ is a simple $U(\mathfrak{gl}_{K|L})$ -module with a basis*

$$\{y_{\mathbf{t}}v_I \mid I \text{ is a } \mathbf{t}\text{-semistandard sequence}\}.$$

Let $\lambda \in \Lambda_{K|L}$ with size $|\lambda| = N$. Given any two sequences $I = (i_1, i_2, \dots, i_N) \in \mathbf{I}_{K|L}^{\times N}, J = (j_1, j_2, \dots, j_N) \in \mathbf{I}_{K|L}^{\times N}$, we define an element of $\mathcal{M}_{K|L}|_{q=1}$ by

$$T(I; J) = (-1)^{\alpha(I, J)} \prod_{a=1}^N t_{i_a j_a}, \quad \alpha(I, J) = \sum_{a>b} [i_a]([i_b] + [j_b]).$$

Then $(\mathbb{C}^{K|L})^{\otimes N}$ affords right $\text{GL}(K|L) := \mathcal{M}_{K|L}|_{q=1} \overline{\mathcal{M}}_{K|L}|_{q=1}$ -comodule structure with structure map

$$\delta^{\otimes N} : (\mathbb{C}^{K|L})^{\otimes N} \rightarrow (\mathbb{C}^{K|L})^{\otimes N} \otimes \text{GL}(K|L), \quad v_J \mapsto \sum_{I \in \mathbf{I}_{K|L}^{\times N}} v_I \otimes T(I; J).$$

As a direct summand of $(\mathbb{C}^{K|L})^{\otimes N}$, $L_\lambda^{K|L}|_{q=1}$ is a $\text{GL}(K|L)$ -comodule with structure map

$$\delta(y_{\mathbf{t}}v_I) = \sum_J y_{\mathbf{t}}v_J \otimes a_J P_{\mathbf{t}}(I, J),$$

where I, J are \mathbf{t} -semistandard sequences and a_J is a scalar multiple. The $P_{\mathbf{t}}(I, J)$'s are matrix elements of $L_\lambda^{K|L}|_{q=1}$ which can be written explicitly as follows:

$$(6.6) \quad P_{\mathbf{t}}(I, J) = \sum_{\sigma \in R(\mathbf{t}), \tau \in C(\mathbf{t})} (-1)^{\ell(\tau)} c(I, (\sigma\tau)^{-1}) T(\sigma\tau I, J).$$

We now arrive at the the SFT of invariant theory for $U(\mathfrak{gl}_{m|n})$.

Theorem 6.7. (SFT for $U(\mathfrak{gl}_{m|n})$, [28, Theorem 2.2]) *Let $\Psi_{r|s}^{k|l} : \mathcal{M}_{r|s}^{k|l}|_{q=1} \rightarrow \mathcal{X}_{r|s}^{k|l}|_{q=1}$ be the surjective superalgebra homomorphism.*

- (1) *As a two-sided ideal of $\mathcal{M}_{r|s}^{k|l}|_{q=1}$, $\text{Ker } \Psi_{r|s}^{k|l}$ is generated by $P_{\mathfrak{t}}(I, J)$, where \mathfrak{t} is a fixed standard tableau of shape $\lambda_c = ((m+1)^{n+1})$ and I and J are \mathfrak{t} -semistandard sequences with elements from $\hat{\mathbf{I}}_{k|l}$ and $\hat{\mathbf{I}}_{r|s}$, respectively.*
- (2) *$\Psi_{r|s}^{k|l}$ is an isomorphism if and only if $m \geq \min\{k, r\}$ and $n \geq \min\{r, s\}$. In this case, $\mathcal{X}_{r|s}^{k|l}|_{q=1}$ is a polynomial superalgebra generated by X_{ab} , $a \in \mathbf{I}_{k|l}$, $b \in \mathbf{I}_{r|s}$ subject to relations as in Lemma 6.4, and it has a monomial basis $\{X^{\mathbf{m}} = \prod_{(a,b)}^{\infty} X_{ab}^{m_{ab}} \mid \mathbf{m} \in \mathfrak{M}_{r|s}^{k|l}\}$.*

Proof. By Theorem 5.15, we only need to prove part (1). Note that $P_{\mathfrak{t}}(I, J)$ as matrix elements of T_{λ_c} will be killed by $\pi_{r|s}^{k|l}$ unless I and J are \mathfrak{t} -semistandard sequences with elements from $\hat{\mathbf{I}}_{k|l}$ and $\hat{\mathbf{I}}_{r|s}$, thus part (1) follows from (6.4). \square

APPENDIX A. HIGHEST WEIGHT REPRESENTATIONS OF $U_q(\mathfrak{gl}_{m|n})$

A.1. Basics on $\mathfrak{gl}(m|n)$. We collect some well known results on the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ over \mathbb{C} . This Lie superalgebra can be realised as a set of $(m+n) \times (m+n)$ complex super matrices equipped with the Lie super bracket as given in Section 2 (see [13]). Let \mathfrak{h} be the Cartan subalgebra of $\mathfrak{gl}_{m|n}$ consisting of all diagonal matrices and $\{\epsilon_i\}_{1 \leq i \leq m+n}$ be the basis for \mathfrak{h}^* dual to the basis $\{e_{ii}\}_{1 \leq i \leq m+n}$ of \mathfrak{h} . We denote by $\mathcal{E}_{m|n}$ the $m+n$ -dimensional vector space over \mathbb{R} with a basis $\{\epsilon_i\}_{1 \leq i \leq m+n}$. Alternatively, we may use $\delta_\mu := \epsilon_{i+\mu}$ for $1 \leq \mu \leq n$. We endow the basis elements with a total order \triangleleft , which is called *admissible order* if $\epsilon_i \triangleleft \epsilon_{i+1}$ and $\delta_\mu \triangleleft \delta_{\mu+1}$ for all i, μ . As an example, we will consider the following two orders

$$(A.1) \quad \begin{aligned} \text{natural order } <: & \epsilon_1 < \epsilon_2 < \dots < \epsilon_m < \delta_1 < \delta_2 < \dots < \delta_n, \\ \text{filpped order } \blacktriangleleft: & \delta_1 \blacktriangleleft \delta_2 \blacktriangleleft \dots \blacktriangleleft \delta_n \blacktriangleleft \epsilon_1 \blacktriangleleft \epsilon_2 \blacktriangleleft \dots \blacktriangleleft \epsilon_m. \end{aligned}$$

Fix an admissible order and let $\mathcal{E}_1 \triangleleft \mathcal{E}_2 \triangleleft \dots \triangleleft \mathcal{E}_{m+n}$ be the ordered basis of $\mathcal{E}_{m|n}$. We define a symmetric non-degenerate bilinear form on $\mathcal{E}_{m|n}$ by

$$(A.2) \quad (\epsilon_i, \epsilon_j) = (-1)^\theta \delta_{ij}, \quad (\delta_\mu, \delta_\nu) = -(-1)^\theta \delta_{\mu\nu}, \quad (\epsilon_i, \delta_\mu) = (\delta_\mu, \epsilon_i) = 0,$$

where $\theta \in \{0, 1\}$ such that $(\mathcal{E}_1, \mathcal{E}_1) = 1$.

The set Φ^\triangleleft of roots of $\mathfrak{gl}_{m|n}$ can be realised as a subset of $\mathcal{E}_{m|n}$ with an admissible order \triangleleft . Explicitly, each choice of a Borel subalgebra of $\mathfrak{gl}_{m|n}$ corresponds to a choice of positive roots Φ^\triangleleft , and hence a fundamental system $\Pi^\triangleleft = \{\alpha_1, \alpha_2, \dots, \alpha_{m+n}\}$ of simple roots, where $\alpha_i = \mathcal{E}_i - \mathcal{E}_{i+1}$ for $1 \leq i < m+n$. The Weyl group conjugacy classes of Borel subalgebras correspond bijectively to the admissible ordered bases of $\mathcal{E}_{m|n}$.

Let $\mathfrak{b}^\triangleleft$ be the Borel subalgebra of $\mathfrak{gl}_{m|n}$ corresponding to the admissible order \triangleleft . In the matrix form, we have $\mathfrak{b}^\triangleleft = \text{Span}_{\mathbb{C}}\{e_{ij} \mid \mathcal{E}_i \triangleleft \mathcal{E}_j\}$. Given an (m, n) -hook partition λ , we write $M(\lambda)^\triangleleft = U(\mathfrak{gl}_{m|n}) \otimes_{U(\mathfrak{b}^\triangleleft)} \mathbb{C}_\lambda^\triangleleft$ for the Verma module of the $\mathfrak{b}^\triangleleft$ -highest weight λ^\triangleleft , where $\mathbb{C}_\lambda^\triangleleft$ is the one-dimensional $\mathfrak{b}^\triangleleft$ -module of weight λ^\triangleleft . The Verma module has a unique quotient L_λ^\triangleleft with the highest weight λ^\triangleleft , which appears in the tensor product decomposition of $(V^{m|n})^{\otimes N}$ for some integer $N \in \mathbb{Z}_+$. In particular, under the natural order $<$, the highest weight of $L_\lambda^<$ is

$$\lambda^< = \lambda^\natural = \lambda_1 \epsilon_1 + \dots + \lambda_m \epsilon_m + \langle \lambda'_1 - m \rangle \delta_1 + \dots + \langle \lambda'_n - m \rangle \delta_n$$

as defined in (2.4).

We use odd reflections to translate between various labellings of irreducible modules arising from different admissible orders. Two ordered bases for $\mathcal{E}_{m|n}$ are called *adjacent* if they are identical except for a switch of a neighbouring pair \mathcal{E}_i and \mathcal{E}_{i+1} with $\mathcal{E}_i \in \{\epsilon_1, \epsilon_2, \dots, \epsilon_m\}$ and $\mathcal{E}_{i+1} \in \{\delta_1, \delta_2, \dots, \delta_n\}$. For example, $\{\epsilon_1 \triangleleft \epsilon_2 \triangleleft \delta_1 \triangleleft \delta_2\}$ and $\{\epsilon_1 \triangleleft' \delta_1 \triangleleft' \epsilon_2 \triangleleft' \delta_2\}$ are adjacent ordered bases for $\mathcal{E}_{2|2}$. For two adjacent ordered bases which differ at i and $i+1$, the corresponding positive root systems are related by

$$(\Phi^{\triangleleft'})^+ = \{(\Phi^\triangleleft)^+ \setminus \{\alpha_i\}\} \cup \{-\alpha_i\} \quad \text{with} \quad \alpha_i = \mathcal{E}_i - \mathcal{E}_{i+1}.$$

It was discovered by Serganova that

$$\lambda^{\triangleleft'} = \begin{cases} \lambda^{\triangleleft}, & (\lambda, \alpha_i) = 0, \\ \lambda^{\triangleleft} - \alpha_i, & (\lambda, \alpha_i) \neq 0. \end{cases}$$

In particular, we can apply a sequence of odd reflections to the natural order $<$ and then obtain the flipped order \blacktriangleleft , yielding

Proposition A.1. *The $\mathfrak{b}^{\blacktriangleleft}$ -highest weight $\lambda^{\blacktriangleleft}$ for $L_{\lambda}^{\blacktriangleleft}$ is given by*

$$\lambda^{\blacktriangleleft} = \lambda'_1 \delta_1 + \cdots + \lambda'_n \delta_n + \langle \lambda_1 - n \rangle \epsilon_1 + \cdots + \langle \lambda_m - n \rangle \epsilon_m.$$

We use the pair $(\mathfrak{gl}_{m|n}, \Pi^{\triangleleft})$ to denote the general Lie superalgebra $\mathfrak{gl}_{m|n}$ furnished with fundamental system Π^{\triangleleft} arising from admissible order \triangleleft . Then it is worthy to note that $(\mathfrak{gl}_{m|n}, \Pi^{\blacktriangleleft}) = (\mathfrak{gl}_{n|m}, \Pi^{\triangleleft})$, and Proposition A.1 implies that the irreducible $(\mathfrak{gl}_{m|n}, \Pi^{\triangleleft})$ -module $L_{\lambda}^{\triangleleft}$, viewed as $(\mathfrak{gl}_{n|m}, \Pi^{\triangleleft})$ -module, has the highest weight $(\lambda')^{\natural}$ as given in Proposition A.1.

For our purpose, we shall determine the lowest weight of $L_{\lambda}^{\triangleleft}$. We consider the following fundamental system $\Pi^{\mathbb{L}}$

$$\Pi^{\mathbb{L}} := \{\delta_n - \delta_{n-1}, \delta_{n-1} - \delta_{n-2}, \dots, \delta_1 - \epsilon_m, \epsilon_m - \epsilon_{m-1}, \dots, \epsilon_2 - \epsilon_1\},$$

which is conjugate to Π^{\blacktriangleleft} under Weyl group of $\mathfrak{gl}_{m|n}$. Precisely, let w_k be the longest element in the symmetric group Sym_k such that $w_k(i) = k+1-i$, $1 \leq i \leq k$ for any $k \in \mathbb{Z}_+$. Then $\Pi^{\mathbb{L}} = w_m w_n \Pi^{\blacktriangleleft}$, where w_m acts on all ϵ_i and w_n acts on all δ_μ by permuting subscripts. The corresponding Borel subalgebra $\mathfrak{b}^{\mathbb{L}}$, which is conjugate to $\mathfrak{b}^{\blacktriangleleft}$, consists of all lower triangular $(m+n) \times (m+n)$ -super matrices. Therefore, the $\mathfrak{b}^{\triangleleft}$ -lowest weight of $L_{\lambda}^{\triangleleft}$ is the $\mathfrak{b}^{\mathbb{L}}$ -highest weight. By Proposition A.1, we immediately have

Proposition A.2. [30] *The $\mathfrak{b}^{\triangleleft}$ -lowest weight, denoted by $\bar{\lambda}^{\natural}$, of $L_{\lambda}^{\triangleleft}$ is given by*

$$\begin{aligned} \bar{\lambda}^{\natural} &= \langle \lambda_m - n \rangle \epsilon_1 + \cdots + \langle \lambda_1 - n \rangle \epsilon_m + \lambda'_n \delta_1 + \cdots + \lambda'_1 \delta_n \\ &= (\langle \lambda_m - n \rangle, \langle \lambda_{m-1} - n \rangle, \dots, \langle \lambda_1 - n \rangle; \lambda'_n, \lambda'_{n-1}, \dots, \lambda'_1). \end{aligned}$$

A.2. The quantum supergroup $U_q(\mathfrak{gl}_{m|n}, \Pi^{\triangleleft})$. Fix the fundamental system $\Pi^{\triangleleft} = \{\alpha_1, \alpha_2, \dots, \alpha_{m+n}\}$ for $\mathfrak{gl}_{m|n}$ associated with an admissible order \triangleleft , where $\alpha_i = \mathcal{E}_i - \mathcal{E}_{i+1}$ for $1 \leq i < m+n$. Let $\Theta \subseteq \{1, 2, \dots, m+n\}$ be the labelling set of the odd simple roots, i.e., $\{\alpha_s \mid s \in \Theta\}$ is the subset of Π^{\triangleleft} consisting of the odd simple roots. We stick to the nondegenerate bilinear form in (A.2), and define the Cartan matrix of $\mathfrak{gl}_{m|n}$ associated to Π^{\triangleleft} by

$$A = (a_{ij}) \quad \text{with} \quad a_{ij} = \begin{cases} \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, & \text{if } (\alpha_i, \alpha_i) \neq 0, \\ (\alpha_i, \alpha_j), & \text{if } (\alpha_i, \alpha_i) = 0. \end{cases}$$

Definition A.3. The quantum general linear supergroup $U_q(\mathfrak{gl}_{m|n}, \Pi^{\triangleleft})$ with the fundamental system Π^{\triangleleft} is the unital associative \mathcal{K} -superalgebra on generators e_i, f_i ($i = 1, 2, \dots, m+n-1$) and $K_a^{\pm 1}$ ($a = 1, 2, \dots, m+n$), where $e_s, f_s, s \in \Theta$ are odd and the rest are even, subject to the relations

- (R1) $K_a K_a^{-1} = K_a^{-1} K_a = 1, \quad K_a K_b = K_b K_a;$
- (R2) $K_a e_i K_a^{-1} = q^{(\mathcal{E}_a, \mathcal{E}_i - \mathcal{E}_{i+1})} e_i, \quad K_a f_i K_a^{-1} = q^{-(\mathcal{E}_a, \mathcal{E}_i - \mathcal{E}_{i+1})} f_i;$
- (R3) $e_i f_j - (-1)^{[e_i][f_j]} f_j e_i = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_{i+1} K_i^{-1}}{q_i - q_i^{-1}},$ with $q_i = q^{(\mathcal{E}_i, \mathcal{E}_i)};$
- (R4) Serre relations:

$$\begin{aligned} (e_s)^2 &= (f_s)^2 = 0, & \text{if } a_{ss} &= 0, \\ \text{Ad}_{e_i}^{1-a_{ij}}(e_j) &= \text{Ad}_{f_i}^{1-a_{ij}}(f_j) = 0, & \text{if } a_{ii} \neq 0, i \neq j; \end{aligned}$$

(R5) High order Serre relations:

$$\mathrm{Ad}_{e_s} \mathrm{Ad}_{e_{s-1}} \mathrm{Ad}_{e_s}(e_{s+1}) = 0, \quad \mathrm{Ad}_{f_s} \mathrm{Ad}_{f_{s-1}} \mathrm{Ad}_{f_s}(f_{s+1}) = 0, \quad \forall s \in \Theta,$$

where $\mathrm{Ad}_{e_i}(x)$ and $\mathrm{Ad}_{f_i}(x)$ are defined respectively by

$$\begin{aligned} \mathrm{Ad}_{e_i}(x) &= e_i x - (-1)^{[e_i][x]} K_i K_{i+1}^{-1} x K_{i+1} K_i^{-1} e_i, \\ \mathrm{Ad}_{f_i}(x) &= f_i x - (-1)^{[f_i][x]} K_{i+1} K_i^{-1} x K_i K_{i+1}^{-1} f_i. \end{aligned}$$

We refer to [32, 33] for more details on $U_q(\mathfrak{gl}_{m|n}, \Pi^<)$. In particular, Definition 2.1 is the special case with fundamental system $\Pi^<$ and $\Theta = \{m\}$, $e_i = E_{i,i+1}$, $f_i = F_{i+1,i}$. Clearly, $U_q(\mathfrak{gl}_{m|n}, \Pi^<) = U_q(\mathfrak{gl}_{n|m}, \Pi^<)$.

When q is an indeterminate, the finite dimensional irreducible representations of $U_q(\mathfrak{gl}_{m|n})$ are similar to those of $\mathfrak{gl}(m|n)$ [35]. Here we only mention the results in Section A.1 using quantum language. We write $U_q(\mathfrak{gl}_{m|n}) = U_q(\mathfrak{gl}_{m|n}, \Pi^<)$ for short.

Proposition A.4. *Let $\lambda \in \Lambda_{m|n}$ be an (m, n) -hook partition and $L_\lambda^{m|n}$ be the irreducible $U_q(\mathfrak{gl}_{m|n})$ -module with the highest weight $\lambda^\natural \in \Lambda_{m|n}^\natural$ for any $m, n \in \mathbb{Z}_+$.*

- (1) *The $U_q(\mathfrak{gl}_{m|n})$ -module $L_\lambda^{m|n}$ has the highest weight $(\lambda')^\natural$ when it is viewed as $U_q(\mathfrak{gl}_{n|m})$ -module, that is, $L_\lambda^{m|n} \cong L_{\lambda'}^{n|m}$ as $U_q(\mathfrak{gl}_{n|m})$ -modules.*
- (2) *The lowest weight $\bar{\lambda}^\natural$ of $U_q(\mathfrak{gl}_{m|n})$ -module $L_\lambda^{m|n}$ is given by the formula*

$$\bar{\lambda}^\natural = (\langle \lambda_m - n \rangle, \langle \lambda_{m-1} - n \rangle, \dots, \langle \lambda_1 - n \rangle; \lambda'_n, \lambda'_{n-1}, \dots, \lambda'_1).$$

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REFERENCES

- [1] A. Berele and A. Regev, “Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras”, *Adv. Math.* **64** (1987), 118-175.
- [2] A. Berenstein and S. Zwicknagl, “Braided symmetric and exterior algebras”. *Trans. Amer. Math. Soc.* **360** (2008), 3429-3472.
- [3] S. Cautis, J. Kamnitzer and S. Morrison, “Webs and quantum skew Howe duality”. *Math. Ann.* **360** (2014), 351-390.
- [4] S.-J. Cheng and W. Wang, “Howe duality for Lie superalgebras”, *Compositio Math.* **128** (2001), 55-94.
- [5] S.-J. Cheng, W. Wang and R.B. Zhang, “Super duality and Kazhdan-Lusztig polynomials”. *Trans. Amer. Math. Soc.* **360** (2008) 5883-5924.
- [6] C. De Concini, D. Eisenbud and C. Procesi, “Young diagrams and determinantal varieties”. *Invent. Math.* **56** (1980), 129-165.
- [7] J. Du and H. Rui, “Quantum Schur superalgebras and Kazhdan-Lusztig combinatorics”. *J. Pure Appl. Algebra* **215** (2011), 2715-2737.
- [8] K. R. Goodearl and T. H. Lenagan, “Quantum determinantal ideals”, *Duke Math. J.* **103** (2000), 165-190.
- [9] K. R. Goodearl, T. H. Lenagan and L. Rigal, “The first fundamental theorem of coinvariant theory for the quantum general linear group”, *Publ. Res. Inst. Math. Sci.* **36** (2000), 269-296.
- [10] P. H. Hai, “Realizations of quantum hom-spaces, invariant theory, and quantum determinantal ideals”. *J. Algebra* **248** (2002), 50-84.
- [11] R. Howe, “Remarks on classical invariant theory”. *Trans. Amer. Math. Soc.* **313** (1989), 539-570.
- [12] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, The Schur lectures, Israel Math. Conf. Proc. 8, (Tel Aviv), (1992), 1-182.
- [13] V. Kac, “Lie superalgebras”, *Adv. Math.* **26** (1977), 8-96.
- [14] G. Lusztig, “On a theorem of Benson and Curtis”, *J. Algebra* **71** (1981), 490-498.
- [15] G. I. Lehrer and R. B. Zhang, “The first fundamental theorem of invariant theory for the orthosymplectic supergroup”, *Comm. Math. Phys.* (2016), 1-42.
- [16] G. I. Lehrer, H. Zhang and R. B. Zhang, “A quantum analogue of the first fundamental theorem of invariant theory of classical invariant theory”, *Comm. Math. Phys.* **301** (2011), 131-174.

- [17] G. I. Lehrer, H. Zhang and R. B. Zhang, “First fundamental theorems of invariant theory for quantum supergroups”. arXiv: 1602.04885.
- [18] H. Zhang and R.B. Zhang, “Dual canonical bases for the quantum general linear supergroup”. *J. Algebra* **304** (2006), 1026-1058.
- [19] H. Mitsuhashi, “Schur-Weyl reciprocity between the quantum superalgebra and the Iwahori-Hecke algebra”. *Algebr. Represent. Theory* **9** (2006), 309-322.
- [20] Y. Manin, “Multiparametric quantum deformation of the general linear supergroup”. *Comm. Math. Phys.* **123** (1989), 163-175.
- [21] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Reg. Conf. Ser. Math., vol. 82, American Mathematical Society, Providence, RI, 1993.
- [22] M. Noumi, H. Yamada and K. Mimachi, “Finite dimensional representations of the quantum group $GL_q(n; \mathbb{C})$ and the zonal spherical functions on $U_q(n-1) \backslash U_q(n)$ ”. *Japan. J. Math. (N.S.)* **19** (1993), 31-80.
- [23] H. Queffelec and A. Sartori, “Mixed quantum skew Howe duality and link invariants of type A”. arXiv: 1504.01225.
- [24] D. Rose and D. Tubbenhauer, “Symmetric webs, Jones-Wenzl recursions, and q -Howe duality”. *Int. Math. Res. Not.* (2016), 5249-5290.
- [25] N. Yu. Reshetikhin, L. A. Takhtadzhyan and L. D. Faddeev, “Quantization of Lie groups and Lie algebras”. *Algebra i Analiz* **1** (1989), 178-206.
- [26] E. Strickland, “Classical invariant theory for the quantum symplectic group”. *Adv. Math.* **123** (1996), 78-90.
- [27] A. Sergeev, “An analog of the classical invariant theory for Lie superalgebras. I”, *Michigan Math. J.* **49** (2001), 113-146.
- [28] A. Sergeev, “An analog of the classical invariant theory for Lie superalgebras. II”, *Michigan Math. J.* **49** (2001), 147-168.
- [29] M. Scheunert and R. B. Zhang, “The general linear supergroup and its Hopf superalgebra of regular functions”. *J. Algebra* **254** (2002), 44-83.
- [30] J. Van der Jeugt and R.B. Zhang, “Characters and composition factor multiplicities for the Lie superalgebra $gl(m/n)$ ”. *Lett. Math. Phys.* **47** (1999) 49-61.
- [31] Y. Wu and R. B. Zhang, “Unitary highest weight representations of quantum general linear superalgebra”, *J. Algebra* **321** (2009), 3568-3593.
- [32] Y. Xu and R. B. Zhang, “Quantum correspondences of affine Lie superalgebras”. arXiv:1607.01142.
- [33] H. Yamane, “Quantized enveloping algebras associated with simple Lie superalgebras and their universal R-matrices”. *Publ. Res. Inst. Math. Sci.* **30** (1994), 15-87.
- [34] R.B. Zhang, “Universal L -operator and invariants of the quantum supergroup $U_q(gl_{m|n})$ ”. *J. Math. Phys.* **33** (1992), 1970-1979.
- [35] R.B. Zhang, “Finite dimensional irreducible representations of the quantum supergroup $U_q(gl_{m|n})$ ”. *J. Math. Phys.* **34** (1993), 1236-1254.
- [36] R.B. Zhang, “Structure and representations of the quantum general linear supergroup”. *Comm. Math. Phys.* **195** (1998), 525-547.
- [37] R.B. Zhang, “Howe duality and the quantum general linear group”. *Proc. Amer. Math. Soc.* **131** (2003), 2681-2692.

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